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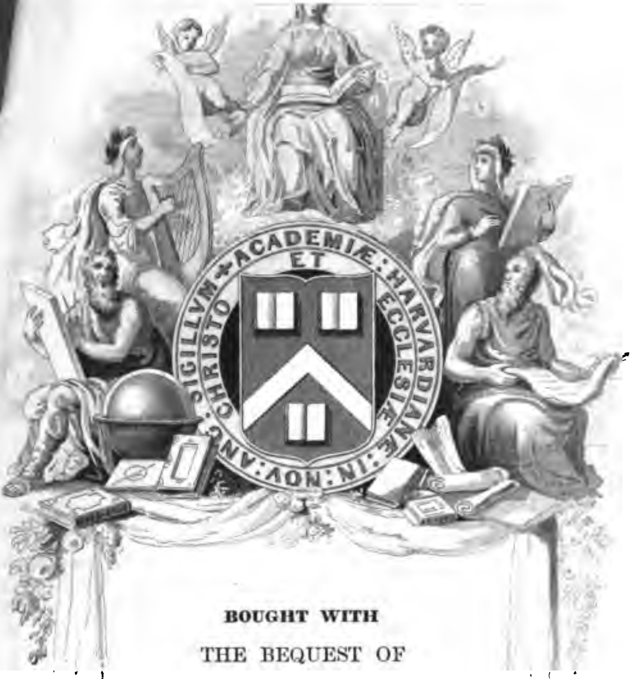

BOUGHT WITH

THE BEQUEST OF

HORACE APPLETON HAVEN,

Of Portsmouth, N. H.

(Class of 1842.)

Rec'd 1 July, 1871.

CAMBRIDGE

LESSONS ON FORM;

OR,

AN INTRODUCTION TO GEOMETRY,

AS GIVEN IN A PESTALOZZIAN SCHOOL,

CHEAM, SURREY.

BY C. REINER,

TEACHER OF MATHEMATICS IN CHEAM SCHOOL.

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PREFACE.

BACON has made an observation to this effect—that a man really possesses only that knowledge, which he in some sort creates for himself. To apply to intellectual instruction the principle implied in these words was the aim of Pestalozzi. It is a principle admitting of various degrees, as well as modes of application, in the different branches of human knowledge; but in no one can it be more extensively applied than in Geometry. That science is peculiarly the creation of the human mind, in which, independent of external nature, and complete in its own resources, it builds up the solid but airy fabric of its abstractions. It needs no laboratory to test its conclusions, no observatory to obtain data for its calculations; rendering aid to other sciences, it asks none for itself.

Hence, that teacher will act most in conformity with the genuine character of the science, and consequently will render the study of it the most interesting and the most improving, who invites and trains his pupils to create the largest portion of it for themselves. In Geometry, the master must not dogmatise, either in his own person or through the medium of his book; but, he must lead his pupils to observe, to determine, to demonstrate for themselves. In order to accomplish this, he must study the intellectual process in the acquisition of *original* mathematical knowledge; and having ascertained what are the conditions of successful investigation, he must so arrange his plan of instruction as that these conditions may be perfectly supplied. He cannot fail to perceive that the leading requisites are a clear apprehension of the subject matter, and well-formed habits of mathematical reasoning. To these must of course be added a familiar acquaintance with the science as far as it has been elaborated. The master, led by these considerations, will, in directing the first labours of his pupils, consider it as his especial aim, to enable them to form clear apprehensions of the subject matter of Geometry, and then to develope the power of mathematical reasoning. Aware that clearness of apprehension can take place only when the idea to be formed is proximate to some idea already clearly formed—when the step, which the mind is required to take, is really the next in succession to the step already taken, he will commence his instruction exactly at that point where his pupils already are, and in that manner which best accords with the measure of their development. As his pupils are unaccustomed to pure abstractions, he will not commence with abstract definitions. But supposing them, through the medium of 'Lessons on Objects' to have had their attention directed to the forms which matter assumes, he will present in his first lessons a transition from the promiscuous assemblage of forms to a particular group of them, consisting of the sphere, the cone, the pyramids, the prisms, and the five regular bodies. In conformity with the plan pursued in 'Lessons on Objects,' the pupils will examine these solids, state what they perceive at the first glance, then by more close and attentive examination, directed by the master, discover and supply the deficiencies in their first perception, and afford him an occasion for connecting their new ideas with adequate technical expressions.

The master's next aim must be to cultivate the power of abstract mathematical reasoning. With a view to this end, he may advantageously avail himself of the knowledge, obtained by the pupils from the solids, in the manner above described. Here, then, he will lead them to deduce the necessary consequences from the facts which they know to be true, and then invite them to examine the object and see whether their reasoning has led to a correct result. Thus, if a child has ascertained and knows that two sides of different planes are requisite to form an edge, and that a certain solid (an octahedron) is bounded by eight triangular planes, he will be required to determine from these data the number of edges which that solid has. He will reason thus:—Eight triangular faces have twenty-four sides; two sides form one edge: therefore, as many times as there are two sides in these twenty-four sides, so many edges that body must have,—that is, twelve edges. This result being obtained, the object is presented to him for examination, and he perceives by actual observation the truth of that conclusion at which he had arrived by abstract reasoning.

These lessons form the basis of the Introduction to

Geometry, and their results are, correct ideas of the subject matter of the subsequent lessons, adequate expressions for these ideas, and sound knowledge of the definitions, which form the connecting link between physical and abstract truths.

In the former part of this work, a mode of accomplishing these points is set forth: in the second part, the further development of the power of abstract reasoning is connected with a direct preparation for the study of Euclid's Elements. That work exhibits a series of mathematical reasonings and deductions, arranged in the most perfect logical order, so that the truths demonstrated rest, in necessary sequence, on the smallest possible number of axioms and postulates. But, admirable as it may be in itself, viewed simply in relation to the science, it is not, viewed pædagogically, an elementary work. It is fitted for the matured, and not for the opening mind. The judicious teacher will desire to present to his pupils the subject matter of Euclid in such a mode and in such order as that in studying it the higher faculties of their minds may be most effectually exercised and improved. For thus only can the intellectual food be assimilated to the intellect itself—be received, as it were, into its substance, and nourish, and strengthen, and expand its powers.

These Lessons on Form present *a* mode in which these principles are applied: other modes, perhaps better ones, may be arranged;—we but say with Horace,

> —— Si quid novisti rectius *istis*
> Candidus imperti; si non, *his* utere mecum.

It has been found in the actual use of these Lessons for a considerable period, that a larger average number of pupils are brought to study the Mathematics with decided success, and that all pursue them in a superior manner. There is much less of mere mechanical committing to memory, of mere otiose admission and comprehension of demonstrations ready-made, and proportionably more of independent judgment and original reasoning. They not only learn Mathematics, but they become Mathematicians.

Hence, when Euclid's Elements and the higher branches of Mathematics are to be read, the pupils are found competent to demonstrate for themselves the greater part of the propositions, and have recourse to books only for occasional correction or improvement of their processes, and for fixing more firmly in their memory the results.

These advantages arise from the application of a principle generally neglected in early education, but deserving of attentive consideration and universal adoption; namely, that "Every course of *scientific* instruction should be preceded by a preparatory course, arranged on *psychological* principles." FIRST FORM THE MIND, THEN FURNISH IT.

C. MAYO.

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LESSONS ON FORM,

BEING AN

INTRODUCTION TO GEOMETRY.

LESSON I.

THE master places before his pupils a variety of objects, among which there should be (the following solids, viz.) the five regular solids, viz. several of the prisms and pyramids, the cylinder, the cone, and the sphere.

Master.—I have set these objects before you, that you may find out some properties common to them all. Endeavour to discover them.

(The answer of each pupil should be subjected to the consideration of the class, and be tried if in reality it equally applies to all objects.)

Pupils.— These and all other objects occupy a space.

M.—In how many directions does each extend?

P.—In three directions: in length, in breadth, and in depth.

One of the pupils said, " and in thickness."

M. (*holding up a book.*)—Which would you call the length of this book?—Which is its breadth? Does it extend in another direction?—By what word will you describe it?

P.—Thickness.

M.—Name an object of which it would be proper to say *depth* instead of *thickness.*

P.—A well extends in length, in breadth, and in depth: so does the sea, a pond, a lake, a river.

M.—Objects considered with reference to these three dimensions only are called *solids.* What other property have all solids in common?

P.—They are all bounded by a *surface.*

M. (*holding up a sphere and a prism.*)—In what does the surface of one of these objects chiefly differ from the surface of the other?

P.—The one is composed of several surfaces, and the other is bounded only by one curved surface.

M.—In what does a surface consist?

P.—In extension of length and breadth: a surface is the boundary of anything.

M.—What happens if a surface be removed from an object?

P.—A part of the object is likewise removed by removing a surface.

M.—Is the object, by doing so, increased or decreased?

P.—It is decreased.

M.—In how many directions is it decreased?

P.—It is decreased either in length, or in breadth, or in thickness.

M.—Can a surface exist without the object of which it is a surface? Can you hold a surface in your hand without holding the object itself?

P.—No.

On this question being asked, one of the pupils said, a shadow is a surface existing without a concomitant solid. The master will of course convince his pupils of the error, should a similar answer be given.

M. (*holding up a prism.*)—What is meant by the surface of this object?

P.—The assemblage of the several surfaces which bound it.

M.—If we wish to distinguish one of these several surfaces from the total number of surfaces, it is usual to call it *one of its faces.* Now state what may be said in general of the number of faces by which all objects are bounded.

P.—All objects are bounded either by one face only, or by several faces.

M.—Now examine more minutely the faces of these objects, and class those together which you think to have similar faces.

The master should allow the pupils some time for arranging the objects before him into groups, until they have perceived that they may be classed properly into three distinct groups;—the one comprehending those which are bounded by plane faces; the next, those that are bounded by plane and curved faces; and lastly, those that have only one curved surface.

P.—All these objects are either bounded by *plane* *faces*, or by *plane* and *curved* faces, or by only *one curved* surface.

M.—Now examine the boundaries of the faces of the first group you have mentioned. What do you observe?

P.—They are all *straight* lines.

M.—And the boundaries of the other group?

P.—Curved lines and straight lines, or only curved lines.

The substance of the lesson is then written on the school-slate by the master, and the pupils are required to commit it to memory. Thus:—

1. All objects discernible by the senses are extended in three dimensions: namely, in *length*, in *breadth*, and in *depth* or *thickness*.

2. Objects considered with reference to these three dimensions are called *solids*.

3. The surface of a solid is its length and breadth considered without reference to its depth.

4. Every solid is bounded either by one *surface* only, or by several *faces*.

5. Solids are either bounded by *plane* faces, or by both *plane* and *curved* faces, or by only *curved* faces.

6. The boundaries of faces are either *straight* lines, or both *straight and curved* lines, or only *curved* lines.

When the above is committed to memory, it is effaced from the slate, and the pupils are required to write it from memory and in the same order.

LESSON II.

At the beginning of this and every following lesson, the pupils ought to be required to recapitulate the preceding lesson, first *viva voce*, and then by writing it out on their slates.

M.—We will now first examine those solids which are bounded by plane faces only. See in what respect their faces differ.

P.—In size, in shape, and in the number of straight lines which bound them.

M.—Speaking of the boundaries of faces, it is usual to call them *sides*, instead of lines. I have brought here a considerable number of solids which are bounded by plane faces. Arrange them according to the number of sides by which some of their faces are bounded, beginning with the least. What is the least number of sides by which some of the faces are bounded?

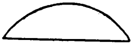
P.—By three straight lines—by three sides.

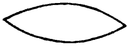
M.—And by what word will you express the space which three straight lines inclose?

P.—A three-sided face—a triangle.

M.—Imitate a triangle on your slates. How many lines are necessary to inclose a space? Try one, two, three. If a space is inclosed by two lines, what sort of lines must these be?

P.—Either a straight and a curved line, or two curved lines.





M.—Which face have you placed next in succession to the triangle?

P.—One which is bounded by four sides.

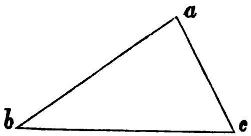
M.—Imitate it on your slates. Which of the faces come next?

P.—The five-sided face; then the six, seven, and eight-sided face.

M.—Imitate all these faces on your slates. Examine the three-sided figure on your slates: in how many points do its three sides meet?

P.—In three points.

M.—(*Draws a triangle upon the school-slate*).



I will put the letters *a*, *b*, *c* at the three points, in order that we may be able to distinguish one side from the others. By what word will you express the position of the line *a b*, to the line *b c*?

P.—The line *a b* is inclined to the line *b c*.

M.—And how many inclinations have the three sides to each other?

P.—Three inclinations.

M.—The inclination which one line has to another line is called *an angle.* How many angles are in a three-sided figure?

P.—Three angles.

M.—See how many angles there are in each of the figures on your slates.

P.—A four-sided figure has four angles; a five-sided, five; a six-sided, six; a seven-sided, seven; and an eight-sided has eight angles.

M.—Can you imagine a figure having nine, ten, eleven, etc. sides? Describe them on your slates, and observe how many angles each figure has.

P.—Every figure has as many angles as it has sides.

M.—You have mentioned another word instead of three-sided figure.

P.—Yes, a triangle.

M.—From what circumstance do you think it is called thus?

P.—From its having three angles.

M.—The names of these several faces are derived sometimes from the number of their angles, and sometimes from the number of their sides. Thus, a three-sided face is sometimes called a *trilateral* figure (from the Latin *tres*, three, and *latus*, a side), or a triangle; a four-sided face is called a *quadrilateral* figure (from the Latin *quatuor*, four, and *latus*, a side); a five-sided face, a *pentagon* (from the Greek πέντε, five, and γωνία, angle); a six-sided face, a *hexagon*, (from the Greek ἕξ, six, and γωνία, angle); a seven-sided face, a *heptagon* (from the Greek ἕπτα, seven, and γωνία, angle); an eight-sided face, an *octagon* (from the Greek ὀκτὼ, eight, and γωνία, angle). And if this mode of expression be extended to faces which are bounded by many sides, they are called *polygons*, (from the Greek πολυς many, and γωνία, angle).

As before, the pupils are called upon to reproduce the lesson on their slates; the substance of which is then arranged into sentences, and written by the master on the large school-slate, the pupils committing them to memory.

1.—Solids bounded by plane faces differ in shape and in the number of their faces.

2.—Their faces differ in the number of their sides.

3.—A face bounded by three sides is called *trilateral;* by four sides, *quadrilateral;* by five sides, a *pentagon;* by six sides, a *hexagon;* by seven sides, a *heptagon;* by eight sides, an *octagon;* by nine or more sides, a *polygon.*

4.—An *angle* is the inclination of two lines to one another which meet in a point.

5.—A trilateral face has three angles, it is therefore called a *triangle;* a quadrilateral has four angles; a pentagon has five, a hexagon six, a heptagon seven, an octagon eight; a polygon has as many angles as it has sides.

LESSON III.

M.—What other parts do you discover on these solids?

P.—Corners and edges.

M.—How are the corners formed?

P.—By several angles of different planes meeting in one point; or by several edges meeting in one point.

M.—How many edges or angles of different faces are at least required to form a corner or *solid angle?* Try, one—two—three.

P.—Three at least.

M.—Instead of "corners," say *solid angles;* how are the edges formed?

P.—By the meeting of two faces.

DESCRIPTION OF THE FIVE REGULAR SOLIDS.*

M.—Which of these five solids is bounded by the least number of faces? By how many faces is it bounded? This solid is therefore called *Tetrahedron* (from the Greek *τετρα*, four, and *ἕδραι*, seats).

M.—What are the four faces?

P.—Four triangles.

M.—How many sides have four triangles?

P.—Twelve.

M.—How many of these sides are there to each of the edges?

P.—Two sides.

---

[*] Tetrahedron, Hexahedron Octahedron, Dodecahedron, Icosahedron.

M.—How many edges therefore must this solid have?

P.—Six edges; because there are six twos in twelve.

M.—Now take the solid, examine it, and see whether it is so.—How many angles are there about each corner or solid angle?

It is important that the pupils be convinced by actual examination of the solid, that the calculation which they have made is strictly true.

P.—Three angles.

M.—These angles are called *plane angles;* can you tell why?

P.—Because they are the angles of the plane faces.

M.—How many plane angles are there in the four triangular faces?

P.—Twelve plane angles.

M.—How many solid angles must the tetrahedron have?

P.—Four solid angles; because about every solid angle there are three plane angles, and there are four threes in twelve.

M.—See whether it is so.

SUBSTANCE OF THE LESSON.

1.—Two faces meeting laterally form an edge.

2.—Three or more edges meeting in one point form a solid angle.

3.—The tetrahedron is a solid bounded by four triangular faces: it has six edges, and four solid angles.

LESSON IV.

M.—Compare the sides of the faces of the tetrahedron. What do you observe?

P.—They are of the same length; they are equal.

M.—How will you call a triangle which has three equal sides?

P.—An equal-sided triangle.

M.—Call it an *equilateral* triangle (from the Latin *æquus*, equal, and *latus*, side). Describe an equi-lateral triangle on your slates; put letters at the angles. —Are all triangles necessarily equi-lateral?

P.—No; for two sides of a triangle may be equal to each other, and the third unequal; or the three sides may be unequal.

The master desires the pupils to draw such triangles upon their slates; after which, he may describe an equilateral, an isosceles, and a scalene triangle upon the school-slate, and, pointing to them, continue.

M.—A triangle having only two of its sides equal to each other is called an *isosceles* (from the Greek ἰσος, equal, and σκέλος, a leg); and the unequal side is called its *base.* And a triangle having none of its sides equal to each other is called a *scalene* (from the Greek σκαζω, to limp, and σκαλήνος, unequal) triangle.—Compare the angles of the faces of the tetrahedron.

P.—They are all equal to each other.

M.—How will you call a triangle of which the angles are equal to each other?

P.—Equiangular triangle.

M.—What are the faces of this solid (showing the octahedron)? By how many such faces is it bounded? How many plane angles are there about each solid angle?

P.—It is bounded by eight plane equilateral and equiangular triangles. There are four plane angles about each of its solid angles.

M.—The name of this solid is octahedron (from the Greek *οκτὼ*, eight, and *ἕδρα*, a seat). Can you find out how many edges the octahedron has, without actually looking at the solid?

P.—It must have twelve edges; because, since it is bounded by eight triangles, there are twenty-four sides to them, two of which belong to each edge; consequently the solid must have twelve edges.

M.—And how many solid angles has the octahedron?

P.—It must have six solid angles; because in its eight faces there are twenty-four plane angles, four of which are about each solid angle, and there are six fours in twenty-four; consequently the octahedron must have six solid angles.

M.—See whether it is so.—What, then, is sufficient to observe in a solid, in order to ascertain the other parts?

P.—It is sufficient to know the number and kind of faces by which the solid is bounded, and also the number of plane angles which are about each of its solid angles.

SUBSTANCE OF THE LESSON.

1.—A triangle having equal sides is called an *equilateral* triangle.

2.—A triangle having equal angles is called an *equi-angular* triangle.

3.—A triangle having two of its sides equal is called an *isosceles* triangle: the unequal side is called its *base.*

4.—A triangle having unequal sides is called a *scalene* triangle.

5.—The faces of the tetrahedron are equilateral and equiangular triangles.

6.—The octahedron is a solid bounded by eight equilateral and equiangular triangles: there are four plane angles about each of its solid angles.

7.—The number and kind of faces, and also the number of plane angles, being known, the number of its edges and solid angles can be ascertained therefrom.

LESSON V.

M.—Is there another among these solids which is bounded by triangles? What is their number? It is therefore called *icosahedron* (from the Greek *εἴκοσι*, twenty, and *ἕδρα*, a seat).—See how many plane angles there are about each of its solid angles, and then calculate the number of its edges and solid angles.

P.—It is bounded by twenty equilateral and equiangular triangles; there are five plane angles about each of its solid angles: it has thirty edges, and fifteen solid angles.

M.—See whether it is so. Which of the remaining two solids is bounded by quadrilateral faces? What is their number? It is therefore called *hexahedron* (from the Greek ἕξ, six, and ἕδρα, a seat), or cube (from the Greek κύϐος, a cube).—How many plane angles are there about each of its solid angles? Calculate the number of its edges and solid angles.

P.—The hexahedron, or cube, is a solid bounded by six quadrilateral faces; three plane angles are about each of its solid angles: it has twelve edges and eight solid angles.

M.—Compare the sides and the angles of the faces of the hexahedron.

P.—The sides are all equal to one another: the angles are likewise equal.

M.—Represent such a face on your slate. The angles of this figure are called *right angles.* Draw a right angle on your slate.—How many right angles can you make with two lines?

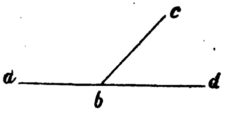
P.—Either one right angle, or two, or four.

M.—Compare these two angles with each other.

P.—They are equal.

M.—Are the two angles which one line makes with another line always equal?

P.—No.

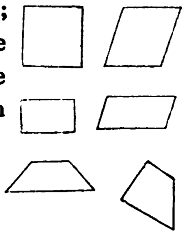


M. When one line makes two angles with another line, these are said to be *adjacent* (from the Latin *ad*, near, and *jacens*, lying) angles. When these adjacent angles are equal, each of them is called a *right* angle.—If these adjacent angles are not equal, what can be said of them?

P.—One is greater than a right angle, the other is less.

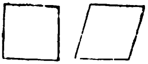
M.—An angle which is greater than a right angle is called *obtuse* (from the Latin *obtusus*, blunted) angle; the angle which is less than a right angle is called *acute* (from the Latin *acutus*, pointed) angle.—Describe on your slates right angles, obtuse, and acute angles. Describe as many different quadrilateral figures as you are able, and first let their difference consist in their sides.

P.—The four sides may be equal; the opposite sides may be equal; the adjacent sides may be equal; three sides may be equal, and the fourth unequal; all four may be unequal.



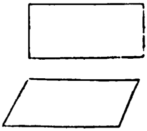
M.—Can you describe two quadrilateral figures having equal sides, and yet being different?

P.—Yes: in the one, all the angles may be equal; in the other, the sides may be equal, but its angles not all equal,—only two pairs of opposite angles are equal.



M.—A quadrilateral figure whose sides are equal, and angles right angles, is called a square. A quadrilateral figure having equal sides, but whose angles are not equal, is called a *rhomb* (from the Latin *rhombus*).—Can you describe two quadrilateral figures having their opposite sides in each equal, and yet be different?

P.—Yes: the angles in the one may all be equal,—that is, right angles; in the other they are not all equal,—only those which are opposite each other.

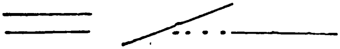


M.—What do you observe respecting the distance of those opposite and equal lines?

P.—Their distance is everywhere the same.

M.—Draw two such lines upon your slates, and two others whose distance is not everywhere the same.

P.—



M.—What happens if the last two lines be lengthened?

P.—They will cross each other.

M.—And what happens if the first two lines be lengthened?

P.—They will not cross each other.

M.—Such lines are called *parallel* (from the Greek παρα, beside, and αλληλων, each other) *lines*, and hence these figures are called *parallelograms* (from the Greek παράλληλος, and γραμμα, a figure); the former a *rectangular parallelogram*, or simply *rectangle*, and the latter merely *parallelogram*. The other quadrilateral figures of which three sides only are equal, or of which all four are unequal, are called *trapeziums* (from the Greek τραπέζιον, a small table).

SUBSTANCE OF THE LESSON.

1.—The icosahedron is a solid bounded by twenty equilateral and equiangular triangles: there are five plane angles about each of its solid angles.

2.—The hexahedron, or cube, is a solid bounded by six squares: there are three plane angles about each of its solid angles.

3.—When one line standing on another line makes the adjacent angles equal to one another, each of them is called a *right* angle.

4.—An angle which is greater than a right angle is called an *obtuse* angle.

5.—An angle which is less than a right angle is called an *acute* angle.

6.—A quadrilateral figure which has equal sides, and its angles right angles, is called a *square*.

7.—A quadrilateral figure which has equal sides and two pairs of equal opposite angles is called a *rhomb*.

8.—A quadrilateral figure which has two pairs of equal opposite and parallel sides, and its angles right angles, is called a *rectangle*.

9.—A quadrilateral figure which has two pairs of equal opposite and parallel sides, but its angles not right angles, is called a *parallelogram*.

10.—All other quadrilateral figures are called *trapeziums*.

LESSON VI.

M.—Examine the last of these five solids.—What are the faces which bound this solid, and what is their number? It is therefore called *dodecahedron* (from the Greek *δώδεκα*, twelve, and *ἕδρα*, a seat).—How many plane angles are there about each of its solid angles? Calculate the number of its edges and solid angles.

P.—The dodecahedron is a solid bounded by twelve pentagons; there are three plane angles about each of its solid angles: it has thirty edges, and twenty solid angles.

M.—Examine the sides and angles of the pentagons.

P.—They are equal.

M.—Are the sides and angles of every pentagon equal?

P.—No; for pentagons may be described of which the sides are unequal, as also the angles.

M.—How will you distinguish the former from the latter?

P.—The former is an equilateral and equiangular pentagon, a regular pentagon; the latter an irregular pentagon.

THE RHOMBOIDAL DODECAHEDRON.

M.—What are the faces of this solid, and what is their number?

P.—The faces are rhombs, and their number is twelve.

M.—How, then, will you call this solid?

P.—Dodecahedron.

M.—How will you distinguish this dodecahedron from the former?

P.—This may be called *rhombic* or *rhomboidal* dodecahedron; the former, *pentagonal* dodecahedron.

M.—How many plane angles are there about each of its solid angles?

P.—About some of its solid angles there are three plane angles; about others, four.

M.—Compare these angles. Do you observe some difference respecting them?

P.—Those of which there are three about a solid angle are greater than those of which there are four.

M.—Define these angles more strictly.

P.—Some of the solid angles are formed by three *obtuse* angles; others, by four *acute* angles.

M.—How many of the angles of a rhomb are obtuse? how many are acute?

P.—In a rhomb there are two obtuse and two acute angles.

M.—How are they situated?

P.—They are opposite each other.

M.—How many obtuse angles are there in all the faces of the rhomboidal dodecahedron, and how many of them are acute angles?

P.—There are twenty-four obtuse and twenty-four acute angles in all the faces.

M.—How many solid angles, then, has this solid? and how many of each sort?

P.—There must be eight solid angles formed by the obtuse angles, and six solid angles formed by the acute angles;—in all, fourteen solid angles.

M.—See whether it is so. By what name will you distinguish each sort of solid angles?

P.—Those formed by the obtuse angles may be called obtuse solid angles; the others, acute.

M.—How many edges has this solid?

P.—Twelve edges.

This and some of the following lessons being dependant on the results of the preceding, they may be considered as a repetition and application of the former. It is therefore not necessary to cause the substance of these lessons to be committed to memory.

LESSON VII.

THE BIPYRAMIDAL DODECAHEDRON.

M.—Examine this solid. What are the faces, and what is their number?

P.—The faces are *isosceles* triangles; their number is twelve.

M.—How will you call this solid?

P.—Dodecahedron.

M.—By what name will you distinguish this dodecahedron from the other two? It is usually called *bipyramidal* dodecahedron, from another kind of solids with which you will hereafter become acquainted.—Examine its solid angles.

P.—Some are formed by six plane angles; others, by four.

M.—Endeavour to distinguish these angles the one from the other.

P.—Some of the solid angles are formed by four angles which are *at the bases* of the isosceles triangles: the other solid angles are formed by the angles which are *opposite* to the bases of the isosceles triangles.

M.—How many angles are there at each of these bases of the triangles?

P.—Two angles at each base.

M.—How may are there of this sort in all the faces?

P.—Twenty-four.

M.—How many solid angles formed by the angles at the base, then, has this solid?

P.—Six solid angles.

M.—And how many solid angles formed by the angles which are opposite the base?

P.—Two solid angles.

M.—What, then, is the total number of solid angles of the bipyramidal dodecahedron, and what must be the number of its edges?

P.—It has eight solid angles, and eighteen edges.

M.—See whether it is so.

THE TRAPEZOHEDRON.

M.—Examine this solid. What are its faces?

P.—Its faces are quadrilateral figures—they are trapeziums.

M.—This solid is therefore called *trapezohedron.* What is the number of its faces?

P.—It is bounded by twenty-four trapeziums.

M.—Examine the angles of each trapezium.

P.—Three of the angles are acute angles, the fourth is obtuse.

M.—Examine the solid angles.

P.—Some of them are formed by four acute, the others by three obtuse angles.

M.—Calculate the number of solid angles of each sort, the total number of solid angles, and the number of edges.

P.—It has eighteen acute solid angles, eight obtuse solid angles; the total number is twenty-six solid angles, and the number of edges is forty-eight.

M.—See whether it is so.

LESSON VIII.

THE PYRAMID.

M. (*putting several pyramids before his pupils.*)—Examine these solids. What do you observe in them?

P.—The faces of each are triangles, except one, which is either a quadrilateral figure, or a pentagon, hexagon, &c.

M.—These kind of solids are called *pyramids.* How would you call the unequal face?

P.—The *base* of the pyramid.

M.—What kind of face may the base of a pyramid be?

P.—Any polygon whatever.

M.—How are the triangles situated?

P.—The triangles all meet in one point, which is opposite the base of the pyramid.

M.—That point is called the *summit* of the pyramid. What is the number of triangular faces?

P.—There are as many triangular faces as the base has sides.

M.—What, then, is the number of faces of a pyramid?

P.—As many faces as the base has sides, more one.

M.—Examine the solid angles.

P.—The angles at the base of the pyramid are formed each by three plane angles; and the angle at the summit, by as many plane angles as there are triangles.

M.—What kind of triangles are they?

P.—Isosceles triangles.

M.—What is the number of solid angles in each pyramid?

P.—There are as many solid angles as the base has sides, more one.

M.—If the base of a pyramid is a triangle, by what name will you distinguish it from another whose base is a quadrilateral figure?

P.—The former may be called a *triangular* pyramid; the latter, *quadrangular*.

M.—Calculate the number of faces, edges, and the number of solid angles of a triangular pyramid.

P.—A triangular pyramid has four triangular faces, six edges, and four solid angles. The tetrahedron is a triangular pyramid.

M.—Calculate the number of faces, edges, and solid angles of a quadrangular pyramid.

P.—A quadrangular pyramid has five faces, four triangular faces, and one which is quadrilateral, which is called the base of the pyramid: it has eight edges, and nine solid angles. The octahedron is formed by two quadrangular pyramids, being joined base to base.

M.—Look again at the bipyramidal dodecahedron. What do you now observe?

P.—It is formed by two hexangular pyramids joined base to base.

M.—Thence its name. Give a description of a pyramid whose base is a polygon of twelve sides.

P.—It has thirteen faces, twenty-four edges, and thirteen solid angles.

M.—How many plane angles are there in all the faces, taken together?

P.—Forty-eight plane angles.

M.—What is the number of sides in all the faces?

P.—Forty-eight sides.

LESSON IX.

THE PRISM.

M. (*placing several prisms before his pupils*)—Examine these solids. What do you observe?

P.—Two of the faces are polygons, the others are rectangles or parallelograms.

M.—Such solids are called prisms. Which of the faces would you call the base of the prism?

P.—Either of the polygons may be its base.

M.—Examine the solid angles. How are they formed, and what must be their number in each prism?

P.—Each is formed by three plane angles; their number must be double the number of the sides of the polygon.

M.—In what manner may you distinguish one prism from another?

P.—By its bases. A prism whose bases are *triangles* we would call *triangular;* one which has quadrilateral bases we would call a *quadrangular* prism, and so on.

M.—Enumerate the faces, edges, and solid angles of a quadrangular prism.

P.—It has six faces, twelve edges, and eight solid angles.

M.—Are you not already acquainted with a quadrangular prism? What is its name?

P.—Hexahedron, or cube.

M. (*showing the parallelopiped*)—Compare this prism with the hexahedron.

P.—The faces of the hexahedron are all equal; they are all squares. In this solid, only two faces are squares; the others are rectangles.

M.—This prism is called a *parallelopiped.* Enumerate the faces, edges, and solid angles of a prism whose bases are polygons of twelve sides.

P.—This prism must have fourteen faces, thirty-six edges, and forty-eight solid angles.

LESSON X.

SOLIDS BOUNDED BY CURVED FACES.

THE SPHERE.

M.—Examine this solid. What can you say of it?

P.—It is bounded only by one curved surface.

M.—Imagine a point within this solid, exactly in the middle of it. What may be said of the surface relatively to this point?

P.—It is everywhere equally distant from this point.

M.—This solid is called a *sphere;* the point which is equally distant from every point of the surface, is called the *centre* of the sphere. Imagine a straight line drawn from any point of the surface of a sphere, through its centre, to the opposite surface: such a line is called a *diameter* (from the Greek διὰ, through, and μέτρον, measure).—How many diameters can be drawn in a sphere?

P.—As many as you please.

M.—What may be said, on comparing the diameters of the same sphere?

P.—Diameters of the same sphere are equal.

M.—When are diameters of different spheres equal?

P.—When the spheres themselves are equal.

M.—And, if two spheres are equal, what may be said of their diameters?

P.—Their diameters must also be equal.

M.—Imagine a straight line drawn from any point in the surface of a sphere to its centre: such a straight line is called a *radius* (from the Latin *radius*, a ray). How many *radii* can be drawn in a sphere? —Compare a radius with a diameter of the same sphere.

P.—An indefinite number.—A diameter is just twice as long as a radius;—a radius is only half a diameter.

M.—If two or more spheres are equal, what may be said of their radii? And if the radii of different spheres be unequal, what may be said of the spheres?

P.—If two or more spheres are equal, their radii must be equal; and if the radii of different spheres be unequal, the spheres must be unequal.

SUBSTANCE OF THIS LESSON.

1. A sphere is a solid bounded by one curved surface, which is everywhere equi-distant from a point, within the solid, called the *centre.*

2. A straight line drawn from any point in the surface of a sphere, through the centre, to the opposite surface, is called a *diameter.*

3. A straight line drawn from any point of the surface of a sphere to the centre, is called a *radius*.

4. A diameter of a sphere is double the length of a radius of the same sphere.

5. Diameters and radii of the same sphere, or of equal spheres, are equal.

6. If spheres are equal, their diameters and radii are equal.

7. If spheres are unequal, their diameters and radii are unequal.

8. If the diameters or radii of different spheres are unequal, the spheres themselves are unequal.

LESSON XI.

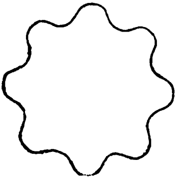
THE CYLINDER.

M.—Examine this solid: it is called a *cylinder*. What can you say of it?

P.—It is bounded by two opposite plane faces, and by one curved face.

M.—Represent the plane faces on your slates. What do you observe?

P.—Each is bounded by one curved line.



M.—Draw, on your slates, a figure, which is also bounded by only one curved line, and yet dissimilar to these figures. Compare, now, these several figures.

P.—The curved line which bounds one of the plane faces of the cylinder is everywhere equally distant from a point within the figure.

M.—Right. And, what would you call this point?

P.—The centre of the figure.

M.—Yes; and the curved line is called the *circumference* of the figure, (from the Latin *circum*, around, and *ferens*, carrying,)—the figure itself being called a *circle.*—How would you define a circle?

P.—A circle is a plane figure, bounded by one curved line everywhere equi-distant from a point within the figure. This point is called, the centre of the circle; and the curved line, the circumference.

M.—A straight line passing through the centre of a circle, and terminated, each way, by the circumference, is called a *diameter;* and a straight line drawn from the centre to the circumference, is called a *radius.*—When circles are equal, what may be said of their diameters, and what of their radii? If circles are unequal, what may be said of their diameters, and what of their radii? If the diameters of several circles be equal, what may be said of these circles? If the radii or diameters of different circles be unequal, what may be said of these circles?

(The pupils are, here, required to illustrate these several cases by drawing circles corresponding with each.)

M.—What happens, if part of a sphere be cut-off?

P.—Each of the parts is a solid, bounded by one curved and one plane surface: the plane surface is a circle.

M.—If several parts be cut-off from a sphere, are the circles obtained by these sections always equal?—When are they equal?

P.—No. If the sphere be cut-through the centre, or if it be cut-through at *equal distances* from the centre,—they are, then, equal.

M.—Imagine a line drawn so as to connect the centres of the circular bases of this cylinder. Examine the angles which this line makes with the faces.

P.—The angles are *right* angles.

M.—And, in this cylinder—what are the angles?

P.—They are *not* right angles.

M.—In the former case, the cylinder is called a *right cylinder;* in the latter case, an *oblique cylinder*, (from the Latin *obliquus*, indirect).

SUBSTANCE OF THIS LESSON.

1. The cylinder is a solid bounded by two opposite plane faces and one curved face.

2. Each of the plane faces is called a *circle.*

3. A circle is a plane figure, bounded by one curved line equi-distant, everywhere, from a certain point within the figure.

4. This point is called the *centre* of the circle.

5. The curved line is called the *circumference.*

6. A straight line passing through the centre, and terminating, each way, in the circumference, is called a *diameter.*

7. A straight line drawn from the centre to the circumference is called a *radius.*

8. When the straight line joining the centres of the circular bases of a cylinder is at *right* angles to either of them, the cylinder is called a *right cylinder;* when the straight line which joins the centres of the circular bases of a cylinder is not at right angles to either of them, the cylinder is called an *oblique cylinder.*

LESSON XII.

THE CONE.

M.—Examine these solids: they are called *cones.* What do you discover?

P.—They are bounded by one plane surface and one curved surface. The plane surface is a circle; the curved surface terminates in a point. This point, in one of the cones, is directly over the centre of the centre: the straight line joining the centre and that point is at *right* angles to the base; in the other cone it is *not* at right angles to the base.

M.—The point, you have mentioned, is called the *summit* or *vertex* of the cone. By what name would you distinguish the one cone from the other?

P.—The one is a *right,* the other an *oblique,* cone.

M.—Compare the *cone* with the other solids. To which of these has it the greatest resemblance?

P.—It is most like the pyramid.

M.—What sort of pyramid would be almost a cone?

P.—That pyramid whose base is a polygon of the *greatest* number of *sides:* the greater the number of the sides of the base, the more nearly does the pyramid approach the cone.

M.—Which of the other solids, besides the pyramid, resembles the cylinder?

P.—The prism.

M.—What sort of prism is most like a cylinder?

P.—That prism whose bases are bounded by the greatest number of sides.

M.—And, which of the solids resembles the sphere?

P.—The *trapezohedron*, or that sort of solid which is bounded by a great number of plane *faces*.

M.—By what name would you describe a solid bounded by a great number of plane faces?

P.—*Polyhedron*—(from the Greek πολὺς, many, and ἕδρα, a seat).

M.—What plane figure, bounded by straight lines, would most nearly resemble the circle?

P.—One bounded by the greatest number of sides.

SURFACES.

CHAPTER I.*

STRAIGHT LINES: ANGLES.

SECTION I.—ONE AND TWO STRAIGHT LINES.

M.—Mention the different parts of a solid.

P.—Solid angles, edges, faces, sides or lines, plane angles.

M.—In this and the following lessons, we shall endeavour to discover the properties of some of these parts. For this purpose we shall begin with the *lines;* and, first, with *one* straight line. Draw *one* straight line, and state all you discover respecting it.

P.—A straight line has length—it *extends* in a certain direction. This is not really a straight line, but a solid; it merely *represents* a straight line—it has two ends or extremities.

M.—What are the extremities of a straight line?

P.—Points.

M.—Tell me, now, what can be done with a straight line.

P.—It can be lengthened, so as to become very long—it can be shortened, so as to become very short: a straight line may be drawn so as to join any two points, but not any *three*, or any *four*, or *more* points.

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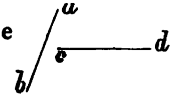
* This division, into *chapters* and *sections*, is adopted, now, for the sake of preserving the *continuity* of the subject. In the course of *tuition*, however, each section may be found to comprise several *lessons*.

M.—Draw *two* straight lines, and state all you can discover from a consideration of them.

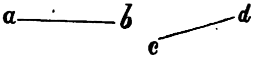
P.—They may be equal in length; they may be unequal in length; they may have the same direction—they may have different directions; they may cross each other.

M.—In order to distinguish the one line from the other, designate the extremities of the one by the letters *a*, *b*; and the extremities of the other by the letters *c*, *d*.—Show me, now, on your slates, the various discoveries you have made respecting *two* straight lines.

P.—1. The line *a b* is *equal* to the line *c d*.



2. The line *a b* is *not* equal to the line *c d*.



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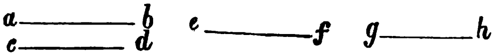
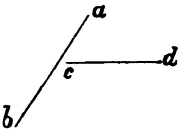


The line *a b* has the same *direction* as the line *c d*.

The line *e f* has, likewise, the same direction as the line *g h*.

4. The line *a b* has *not* the same direction as the line *c d*.



M.—In the first case which you have stated, if the line *a b* is equal to the line *c d*, what happens, if you imagine the line *c d* applied to the line *a b*?

P.—The extremities of the line *cd* may be upon the extremities of the line *a b*, or they may *not*.

M.—How must the line *c d* be applied to the line *a b* so that their extremities shall be upon one another, or shall *coincide*?

P.—One extremity of the line *c d* must be imagined to be upon one of the extremities of the line *a b*; then, the other extremity of *c d* will fall upon the other extremity of *a b*.

M.—In other words, if the straight line *c d* be placed upon the straight line *a b*, so that the point *c* be upon the point *a*, the point *d* of the straight line *c d*——Finish the sentence.

P.—Must fall upon the point *b* of the straight line *a b*.

M.—Why?

P.—Because the line *a b* is *equal* to the line *c d*.

M.—If these straight lines be *not* applied in this manner, will their extremities coincide?

P.—No: their extremities will not be upon one another (will not *coincide*).

M.—In case 2, can the extremities of the lines coincide? Why not?

P.—No; because the lines are *un*equal.

M.—By what means can these lines be made equal?

P.—Either by cutting-off from the longer a part equal to the shorter, or by lengthening the shorter until it becomes equal to the longer.

M.—Instead of "lengthening," say *producing.* In case 3, what may be said of the lines *a b* and *c d*, if they be produced ever so far both ways?

P.—That these lines will never meet.

M.—Straight lines which, when produced ever so far both ways, never meet, are called *parallel lines.* Παράλληλος, from παρὰ, *beside,* and ἀλλήλων, *each other.* What will happen if *e f* and *g h* be produced both ways?

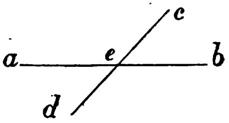
P.—They will meet and form *but one* straight line.

M.—In case 4, what will happen, if *a b* and *c d* be produced both ways?

P.—These lines will meet and will cross each other.

M.—Instead of "cross," say *intersect* each other. The *place where* they intersect is called the *point of intersection.* Draw two lines which shall intersect each other.

P.—*a b* and *c d* intersect each other, at the point *e*, which is, therefore, their point of intersection.

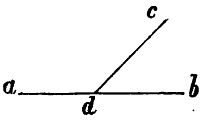


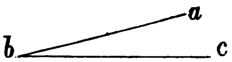
M.—What do straight lines form by intersection?

P.—They form [four] *angles.*

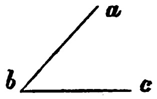
M.—And if two lines *meet,* how many angles do they form?

P.—Either *two* angles or *one* angle.





M.—The point in which the two lines meet is called the *vertex* of the angle; the lines themselves are called the *legs* of the angle. An angle is usually expressed by putting a letter at the extremity of each leg, and at the vertex; and, in naming the angle, the letter which stands at the vertex is inserted between the other two; thus, *c b a*, or *a b c*.—If the legs of an angle be lengthened or shortened, is the angle thereby altered?—Why not?

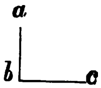


P.—The angle is *not* altered; because the *inclination* of the two lines is *not* altered by either lengthening or shortening the legs.

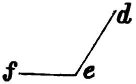
M.—If two lines make an angle, what may that angle be?

P.—Either a right, an obtuse, or an acute angle.

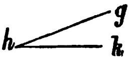
a b c is a right angle;



d e f is an obtuse angle;

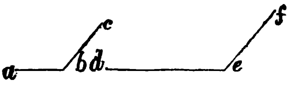


g h k is an acute angle.



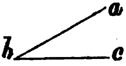
M.—Are *all* obtuse angles equal?

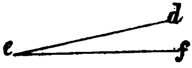
P.—No: *a b c* and *d e f* are *both* obtuse angles; but, *a b c* is, obviously, *less* than *d e f*.



M.—Are *all* acute angles equal?

P.—No; *a b c* and *d e f* are *both* acute angles; but, *a b c* is manifestly *greater* than *d e f.*





M.—Are *all* right angles equal?

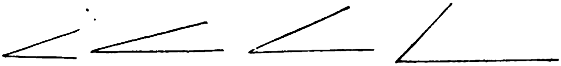
P.—Yes; *all* right angles are equal.

M.—How, then, is it possible to distinguish one obtuse angle from another obtuse angle,—one acute angle from another acute angle?

P.—By comparing each with a right angle. For instance, one obtuse angle may be a right angle and *half* of another right angle; and another obtuse angle may be a right angle and *one-third* of another right angle; one acute angle may be *half* of a right angle, and another acute angle may be *one-third* of a right angle.

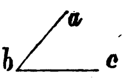
M. Draw several equal and unequal *obtuse* angles. Draw several equal and unequal *acute* angles.

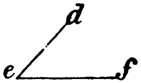
P.—



M.—Make an angle equal to another angle.

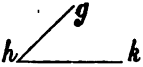
P.—The angle *a b c* is equal to the angle *d e f.*





M.—Make a third angle which is equal to the angle *d e f.*

P.—*g h k* is equal to *d e f.*



M.—Compare the angle *g h k* with the angle *a b c.*

P.—They are equal.

M.—And, if two angles, *together*, are equal to a certain angle, and two other angles are also, *together*, equal to the *same* angle, what may be concluded?

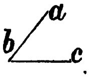
P.—That the first two angles are, together, equal to the second two angles.

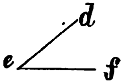
M.—Express this truth generally.

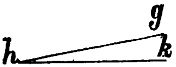
P.—*Angles which are equal to the same angle are equal to one another.*

M.—Make two *equal* angles, and a third angle *not* equal to either.

P.—







M.—To each of the equal angles *add* the angle *g h k:* what are the *sums?*

P.—The angles *a b c* and *g h k*, together, are equal to the angles *d e f* and *g h k*, together.

M.—Express this *generally.*

P.—*If the same angle be added to equal angles, the sums are equal.*

M.—Instead of adding the *same* angle, what might be substituted?

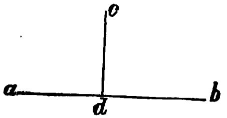
P.—*Equal* angles might be added.

M.—When one line makes two *equal* angles with another line, what are these angles called?

P.—*Right* angles.

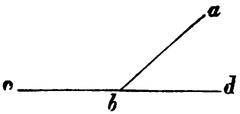
M.—Draw two lines so, that they may be at right angles to each other,—thatis, form right angles.

P.—The angles *a d c* and *b d c* are right angles.



M.—Then, if the *adjacent* angles, formed by the meeting of two straight lines, are equal, what are they called? Draw two straight lines which are *not* at right angles to each other, and name the angles which they form.

P.—The angle *a b c* is *greater* than a right angle—it is an *obtuse* angle; and the angle *a b d* is *less* than a right angle—it is an acute angle.

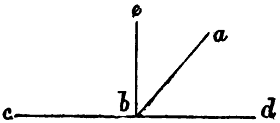


M.—How may you ascertain that the angle *a b c* is not a right angle?

P.—By comparing it with its adjacent angle *a b d*; if the angle *a b c* is *not* equal to its adjacent angle *a b d*, it is not a right angle.

M.—How may you ascertain that the angle *a b c* is *greater* than a *right* angle?

P.—By drawing from the point *b* another line, *b e*, making right angles with *c d*.



M.—By what angle is *a b c greater* than a right angle?

P.—By the angle *a b e.*

M.—And, by what angle is *a b d less* than a right angle?

P.—By the same angle *a b e.*

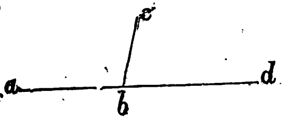
M.—If, then, the two adjacent angles be not right angles, what may be said of the obtuse and the acute angle when compared with one right angle?

P.—The obtuse angle is as much greater than one right angle, as the adjacent angle is less than one right angle.

M.—What, therefore, may be said of the *sum* of the two angles which one straight line makes with another straight line?

P.—The two angles which one straight line makes with another straight line are, together, always equivalent to two right angles.

M.—If the angle *a b c* is one right angle, and one-half of a right angle, that is, $\frac{3}{2}$ right angles, what part of one right angle is its adjacent angle *a b d?*



P.—The angle *a b d* is $\frac{1}{2}$ right angle.

M.—Why?

P.—Because the angles *a b c* and *a b d* are, together, equal to two right angles, (that is, $\frac{4}{2}$ right angles,) and *a b c* is $\frac{3}{2}$ right angles: therefore, *a b c* must be $\frac{1}{2}$ right angle.

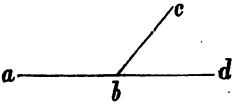
M.—What may be concluded from this example?

P.—That if one angle be known, its adjacent angle, likewise, may be known.

M.—How, then, may you ascertain the *magnitude* of the adjacent angle?

P.—By subtracting the known angle from *two* right angles.

M.—Compare each of the angles *a b c* and *c b d* with two right angles.



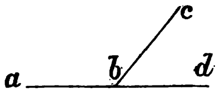
P.—The angle *a b c* is equal to two right angles *minus* [less] the angle *c b d.* The angle *c b d* is equal to two right angles *minus* [less] the angle *a b c.*

M.—We shall, now, substitute *signs* for words frequently occurring; thus, put $\angle$ instead of *angle*, $\angle$s instead of *angles;* $=$ instead of *is or are equal to;* $-$ instead of *less* [minus]; rt. $\angle$ instead of *right angle;* rt. $\angle$s instead of *right angles.*—Now, rewrite the last two sentences on your slates, and make use of these signs.

P.—

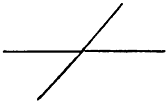
$\angle\, abc = 2$ rt. $\angle$s $- \angle\, cbd$

$\angle\, cbd = 2$ rt. $\angle$s $-\, abc$.



M.—What is the number of angles formed by two straight lines intersecting each other?

P.—Four.



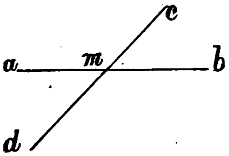
M.—What may be said of them?

P.—They are either four right angles, or they are, together, equal to four right angles.

M.—If one of them is a right angle, what must each of the others be?

P.—Also, a right angle.

M. (*drawing, on the school-slate, two lines intersecting each other.*)—Name two of these angles which, together, are equal to two right angles.



P.—The angles $a\ m\ c$ and $c\ m\ b$.

M. (*writing.*) $\angle$s $a\ m\ c$ and $c\ m\ b = 2$ rt. $\angle$s. Name two other angles which are also equal to two right angles, and let $a\ m\ c$ be one of them.

P.—The angles $a\ m\ c$ and $a\ m\ d$.

M. (*writing.*) $\angle$s $a\ m\ c$ and $a\ m\ d =$, also, 2 rt. $\angle$s: hence, the angles $a\ m\ c$ and $c\ m\ b$ are together equal to—what other two angles?

P.—The two angles $a\ m\ c$ and $a\ m\ d$.

M. (*writing.*)—Hence, $\angle$s $a\ m\ c$ and $c\ m\ b = \angle$s $a\ m\ c$ and $a\ m\ d$. If the angle $a\ m\ c$ be taken away from the former two angles, and, also, from the latter two angles,—what angles remain?

P.—The angles $c\ m\ b$ and $a\ m\ d$.

M.—And, what may be said of these angles?

P.—That they are equal.

M. (*writing.*)—If the angle $a\ m\ c$ be taken from each of these equals, there remains

$$\angle\ c\ m\ b = \angle\ a\ m\ d.$$

How are the angles $c\ m\ b$ and $a\ m\ b$ situated?

P.—They are opposite each other.

M.—For this reason, they are called *opposite* or *vertical* angles. Can you tell me why they are called *vertical*?

P.—Because the vertex of the one angle is, likewise, the vertex of the other.

M.—What other angles are, here, vertical?

P.—The angles *a m c* and *b m d.*

M.—Compare them.

P.—They are equal.

M.—Hence, the opposite or vertical angles formed by the intersection of two straight lines——

P.—Are equal to each other.

M.—This truth we have established; we have *proved* it to be true. If, then, you were called-upon to prove, that, "*if two straight lines intersect each other, the opposite or vertical angles are equal,*" what would you do?

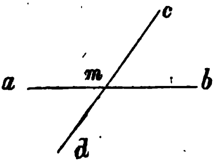
P.—We would do as we have just now done.

M.—Well—the *method* by which a mathematical truth, such as this, is proved, we call its *demonstration.*

M.—I shall rub out what I have written. Demonstrate, each of you, that, "if two straight lines intersect each other, the opposite or vertical angles are equal."

The master will, here, have an opportunity of judging whether what precedes has been clearly understood. It is of importance that the written demonstration should be well performed,—the lines being neatly drawn, &c. The signs, which have been introduced, tend to render the expressions capable of being more quickly revised, and to facilitate the detection of errors. The following is a specimen of the manner of demonstration recommended:—

If two straight lines intersect each other, the opposite or vertical angles are equal.



Let the straight lines $a\,b$ and $c\,d$ intersect each other in the point m; the vertical angles $a\,m\,c$ and $b\,m\,d$, and, also, $c\,m\,b$ and $a\,m\,d$, shall be equal.

Because $\angle$s $a\,m\,c$ and $c\,m\,b = 2$rt. $\angle$s, and, also, the $\angle$s $c\,m\,b$ and $b\,m\,d = 2$ rt. $\angle$s; therefore, $\angle$s $a\,m\,c$ and $c\,m\,b = \angle$s $c\,m\,b$ and $b\,m\,d$: from each of these equals take-away $\angle\ c\,m\,b$; there remains $\angle\ a\,m\,c = b\,m\,d$.

Again,

Because $\angle$s $a\,m\,c$ and $c\,m\,b = 2$ rt. $\angle$s, and, also, the $\angle$s $a\,m\,c$ and $a\,m\,d = 2$ rt. $\angle$s; therefore, $\angle$s $a\,m\,c$ and $c\,m\,b = \angle$s $a\,m\,c$ and $a\,m\,d$: from each of these equals take away $\angle\ a\,m\,c$; there remains $c\,m\,b\ \angle = a\,m\,d$.

M.—On what truth does this demonstration chiefly depend?

P.—That the two angles which one straight line makes with another straight line are together equal to two right angles.

M.—There are two other truths referred-to in your demonstration: what are they?

P.—Angles which are equal to the same angles are equal to each other; and, if the same angle be taken from equal angles, the remaining angles are equal.

M.—Do these last-mentioned truths require demonstration?

P.—No; they are acknowledged at once.

M.—Will it be readily admitted that, the two angles which one straight line makes with another straight line are equal to two right angles; and that, if two straight lines intersect each other, the vertical angles are equal?

P.—No; these truths must be proved; they require *demonstration.*

M.—Truths which are at once *conceded,* and, therefore, do not require demonstration, are called *axioms.* (Greek ἀξίωμα, from ἄξιος, *worthy,*—a proposition *worthy* of being believed.)

M.—Instead of saying, "Angles which are equal to the same angles are equal to each other," we might say, in general, "*Things* which are equal to the *same thing* are equal to each other." Generalize the other axiom in a similar way.

P.—If equals be taken from equals, the remainders are equal.

M.—Endeavour to generalize a similar axiom.

P.—If equals be added to equals, the sums are equal.

M.—From what obvious truth did you deduce this general axiom?

P.—If equal angles be added to equal angles, the *sums* are equal.

The pupils will probably mention several axioms: if not, they may be led to discover some.

It is of importance, that the substance of this section should be reduced to concise sentences, which ought to be written on the school-slate and committed to memory.

SUBSTANCE OF SECTION I.

1. A line is length without breadth.
2. The extremities of a line are *points.*

3. Two straight lines may either be parallel or not.

4. Parallel lines are such, that, if produced both ways ever so far, they do not meet.

5. If *non*-parallel lines be produced far enough, they meet or intersect each other.

6. If two straight lines *meet*, they may either form *one* angle or *two* angles.

7. An angle is the inclination, to one another, of two lines, which meet in one point.

8. That point is called the *vertex* of the angle; and the lines which contain the angle are called its *legs*.

9. Angles may either be equal or unequal.

10. If one straight line standing on another straight line makes the *adjacent* angles *equal* to each other, each of them is called a *right* angle.

11. All right angles are equal to each other.

12. An obtuse angle is that which is greater than a right angle.

13. An acute angle is that which is less than a right angle.

14. All obtuse angles are *not* equal to each other; nor are all acute angles equal to each other.

15. The two angles, which one straight line makes with another straight line, are either two right angles, or they are together *equal* to two right angles.

16. If two straight lines intersect each other, the four angles about the point of intersection are either four right angles, or they are, *together*, *equal* to four right angles.

17. If two straight lines intersect each other, the opposite or vertical angles are equal.

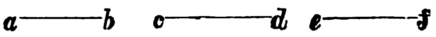
The axioms should, in a similar manner, be written, and learnt by heart.

Frequent repetition of what precedes is indispensably necessary before beginning the next section. The pupils must be able to write the preceding sentences in their *order*, whenever they are so required, and great strictness is to be observed with regard to the demonstration of No. 17.

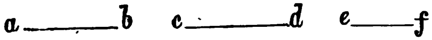
SECTION II.—THREE STRAIGHT LINES.

M.—Draw *three* straight lines, and state what you observe respecting them, proceeding as you did with two straight lines.

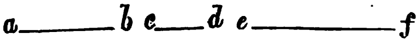
P.—Three straight lines may be equal.



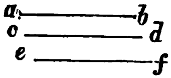
2nd. Two of them may be equal, and the third unequal.



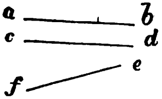
3rd. All three may be unequal.



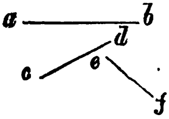
4th. They may all three be parallel.



5th. Two of them may be parallel, and the third non-parallel.



6th. All three may be non-parallel.



M.—In the first case, if the equal straight lines *a b*, *c d*, *e f*, be placed upon each other, so that their extremities *a*, *c*, *e*, coincide, what must happen?

P.—Their other extremities *b*, *d*, and *f*, must likewise coincide; because the lines *a b*, *c d*, and *e f* are equal.

M.—If the line *a b* be added to each of the other lines *c d* and *e f*, what will the sums be?

P.—*a b* together with *c d* must be equal to *a b* together with *e f*.

M.—Why?

P.—Because if equals are added to equals, the *sums* are equal.

M.—Instead of writing, "*a b* together with *c d* is equal to *a b* together with *e f*," we shall make use of a *sign* for the words "together with,"—thus:

$$a\,b + c\,d = a\,b + e\,f.$$

In the *second* case, if the lines *a b*, *c d*, *e f* be placed so that their extremities *a*, *c*, *e*, shall coincide, will their other extremities *b*, *d*, and *f*, likewise coincide?

P.—The extremities *b* and *d* of the equal straight lines *a b* and *c d* will coincide; but the extremity *f* of the *un*equal straight line *e f* will *not* coincide.

M.—If the line *a b* be added to each of the other lines *c d* and *e f*, what will their sums be?

P.—Unequal: for $a\,b + c\,d$ are not equal to $a\,b + e\,f$.

M.—Why not?

P.—Because *a b* and *c d* are equal, and *e f* is unequal; and if equals be added to *un*equals, the sums shall be unequal.

M.—But if the line ef be added to each of the equal lines ab and cd, what will the sums be?

P.—$ab + ef = cd + ef$; because, &c.

M.—In the *third* case, can the unequal lines ab, cd, and ef, be so placed that their *extremities* may coincide?

P.—No; the lines may coincide, but their extremities cannot: because ab, cd, and ef are unequal.

M.—If the line ab be added to each of the other lines, cd and ef, what will the sums be?

P.—The sums must be unequal: because the same line has been added to unequal lines.

M.—In the *fourth* case, can the parallel lines ab, cd, and ef, intersect?

P.—No; because parallel lines are such that, if produced ever so far, they never meet.

M.—If, then, it were discovered that these lines do meet, what must be concluded?

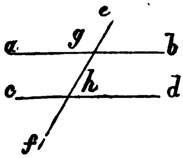
P.—That they are *not* parallel.

M.—In the *fifth* case, since the line ef is not parallel to the parallel lines ab and cd, what will happen if ef be produced?

P.—The line ef will intersect both the parallel lines ab and cd.

M.—Represent this on your slates; put letters at the points of intersection. Which of the angles are equal?

P.—$\left.\begin{array}{l}\angle age = \angle hgb\\ \angle agh = \angle egb\\ \angle chg = \angle fhd\\ \angle chf = \angle ghd\end{array}\right\}$ for, they are vertical angles.



M.—Examine these angles more closely: are others of them equal, besides those you have mentioned?—How is the inclination of the line *a g* to the line *e f* expressed?

P.—By the angle *a g e*, or by the angle *a g f*.

M.—Is there another line which has the same inclination to *e f* which *a g* has to *e f*?

P.—Yes; the line *c h* or *c d*: because *a g* and *c h* are *parallel*.

M.—And, how is that inclination expressed?

P.—By the angle *c h e*, or by the angle *c h f*.

M.—What angles, then, must be equal?

P.—The angle *a g e* must be equal to the angle *c h e*, and the angle *c h f* to the angle *a g h*.

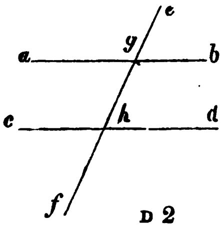
M.—How are the angles *a g e* and *c h e* situated with regard to each other?

P.—The angle *a g e* is *without*—the angle *c h e* is *within*, the parallels.

M.—The angle *a g e* is, therefore, called the *exterior*, and the angle *c h e* the *interior* and *opposite* angle. If, then, two parallel lines intersect another straight line, what may be said of the angles which they form?

P.—If two parallel lines intersect another straight line, they make the exterior angle equal to the interior and opposite angle.

M.—Let *a b* and *c d* be two parallel lines intersecting another straight line, *e f*, in the points *g* and *h*. What are the exterior and the interior and opposite angles?



P.—The exterior $\angle age =$ inter. and oppo. $\angle che$.
" $\angle chf =$ " " $\angle agf$.
" $\angle bge =$ " " $\angle dhg$.
" $\angle fhd =$ " " $\angle bgf$.

M.—To what other angle is $\angle age$ equal?

P.—To the angle bgf; because age and bgf are *vertical* angles.

M.—What must you thence conclude?

P.—That the angle bgf is, also, equal to the angle che.

M.—Angles which are situated as the angles bgf and che, are called *alternate* angles. What angles, besides these, are similarly situated? What are the *alternate* angles?

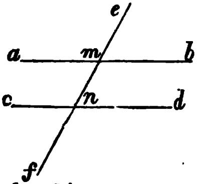
P.—The angles bgf and che are alternate.
" agf and dhe are alternate.

M.—Then, if two parallel lines intersect another straight line, they make the exterior angle equal to the interior and opposite angle, and—— ?

P.—They make the alternate angles equal to each other.

M.—Endeavour, now, to demonstrate this latter truth with as much precision as you have proved No. 17.

P.—Let the parallel lines ab and cd intersect another straight line ef; the alternate angles amf and dne shall be equal, and, also, the alternate angles bmf and cne.



Because the parallel lines ab and cd intersect ef, the exterior $\angle emb =$ interior and opposite $\angle dne$.
But $\angle emb =$ its vertical $\angle amf$;

therefore, $\angle\, a\, m\, f = \angle\, d\, n\, e.$

Again,

Because the parallels *a b* and *c d* intersect the straight line *e f*,

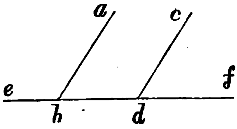
the ext. $\angle\, a\, m\, e =$ int. opp. $\angle\, c\, n\, e$:

But, $\angle\, a\, m\, e =$ its vertical $\angle\, b\, m\, f$;

therefore, $\angle\, b\, m\, f = \angle\, c\, n\, e.$

M.—What axiom did you make use of in this demonstration?

P.—Angles or things which are equal to the same angle or thing, are equal to each other.



M.—Let the parallel lines *a b* and *c d* meet the line *e f* in the points *b* and *d.* Which of these angles would you call *interior* angles?

P.—The angles *a b d* and *c d b* are interior angles.

M.—What angle is equal to the angle *a b d*?

P.—The angle *c d f* is equal to the angle *a b d*; because $\angle\, c\, d\, f$ is the exterior, and $\angle\, a\, b\, d$, the interior and opposite angle.

M.—Hence $\angle\, a\, b\, d + \angle\, c\, d\, b$ is equal to——

P.—$\angle\, c\, d\, f + \angle\, c\, d\, b.$

M.—And what do you know respecting the angles $c\, d\, f + c\, d\, b$?

P.—$\angle\, c\, d\, f + \angle\, c\, d\, b = 2$ rt. $\angle$s.

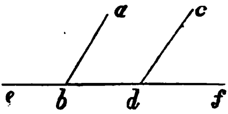
M.—What do you thence conclude?

P.—That the two interior angles, $a\, b\, d + c\, d\, b = 2$ rt. $\angle$s, likewise.

M.—Express, now, in words the truth we have just discovered.

P.—If two parallel lines meet another straight line, they make the two interior angles, together, equal to two right angles.

M.—Demonstrate this truth.



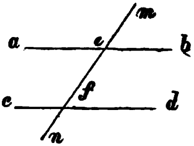
P.—Let the parallel lines *a b*, *c d* meet the line *e f* in the points *b d*, the *two interior* angles *a b d* and *c d b* shall, together, be equal to *two right* angles.

Because the lines *a b* and *c d* are parallel,
the ext. $\angle\, c\,d\,f =$ int. opp. $\angle\, a\,b\,d$.
To each of these angles add $\angle\, c\,d\,b$;
then, $\angle\, c\,d\,f + \angle c\,d\,b = \angle\, a\,b\,d + \angle c\,d\,b$.
But $\angle\, c\,d\,f + \angle c\,d\,b = 2$ rt. $\angle$s;
therefore, $\angle\, a\,b\,d + \angle c\,d\,b =$ likewise, 2 rt. $\angle$s.

M.—What axioms did you use in this demonstration?

P.—"If equals be added to equals, the sums are equal;" and, "things which are equal to the same thing are equal to each other."

M.—Demonstrate this truth from the *equality* of the *alternate* angles. In doing so, you will, of course, use *intersected* lines.



P.—Because *a b* and *c d* are parallel lines, $\angle\, b\,e\,f = \angle\, c\,f\,e$, each being an alternate angle.

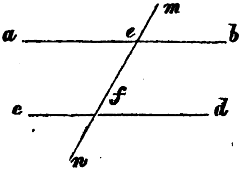
To each of these angles add $\angle\, a\,e\,f$;
then, $\angle\, b\,e\,f + \angle a\,e\,f = \angle\, c\,f\,e + \angle\, a\,e\,f$.
But, $\angle\, b\,e\,f + \angle a\,e\,f = 2$ rt. $\angle$s;
therefore, the 2 int. $\angle$s, $c\,f\,e + a\,e\,f =$ likewise, 2 rt. $\angle$s.

M.—Since the 2 int. $\angle$s, $a\,e\,f + c\,f\,e = 2$ rt. $\angle$s, what must the 2 exterior $\angle$s, $c\,f\,n + a\,e\,m$ be ?

P.—The 2 ext. $\angle$s, $c\,f\,n + a\,e\,m$, must be equal to 2 rt. $\angle$s: because the 2 interior angles, $c\,f\,e + a\,e\,f$, together with the 2 exterior angles, $c\,f\,n + a\,e\,m = 4$ rt. angles, and it is known that the 2 interior $\angle$s, $c\,f\,e + a\,e\,f = 2$ rt. $\angle$s; therefore, the 2 ext. $\angle$s, $c\,f\,n + a\,e\,m$, must be equal to 2 rt. $\angle$s.

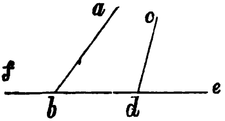
M.—Demonstrate this on your slates.

P.—Because



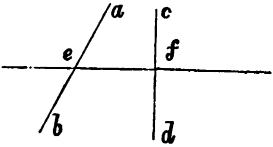
$\angle$s $c\,f\,n + c\,f\,e = 2$ rt. $\angle$s,
and, also, $\angle$s, $a\,e\,m + a\,e\,f = 2$ rt. $\angle$s;
therefore, $\angle$s, $c\,f\,n + c\,f\,e + a\,e\,m + a\,e\,f = 4$ rt. $\angle$s.
But, $\angle$s $c\,f\,e + a\,e\,f = 2$ rt. $\angle$s; because $a\,b$ and $c\,f$ are parallel: therefore,
the 2 ext. $\angle$s, $c\,f\,n + a\,e\,m =$ likewise, 2 rt. $\angle$s.

M.—If it be known that, the exterior angle $c\,d\,e$ is *not* equal to the interior and opposite angle $a\,b\,d$, what must be concluded?

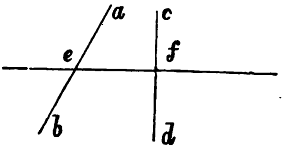


P.—That the lines $a\,b$ and $d\,c$ are not *parallel.*

M.—Again, if it be known that the angles $a\,e\,f$ and $d\,f\,e$ are not equal, what must be concluded?



P.—That the lines *a b* and *c d* are not parallel.



M.—Again, if it be known that the 2 interior angles *a e f* + *c f e* are not equal to 2 rt. ∠s, what must be concluded?

P.—That the lines *a b* and *c d* are not parallel.

M.—If the ∠s, *a e f* + *c f e*, are not equal to 2 rt. ∠s, what may they be, together?

P.—They may be *together* either *more* or *less* than 2 rt. ∠s.

M.—If the angles *a e f* + *c f e* be *less* than 2 rt. ∠s, what will happen if the lines *a b*, *c d*, be produced both ways?

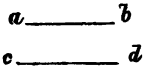
P.—The lines *a b* and *c d* will *meet*, toward *a* and *c*.

M.—If the angles *b e f* + *d f e* be *more* than 2 rt. ∠s, what will happen if the lines *a b* and *c d* be produced both ways?

P.—The lines *a b* and *c d* will *not* meet toward the points *b*, *d*.

M.—Hence, if any two straight lines intersect another straight line, on which side of this line will the two lines meet, if produced far enough?

P.—The lines will meet on that side of the intersecting line where the two interior angles are together less than two right angles.



M.—If, then, we wished to ascertain whether the lines *a b* and *c d* are parallel or not, what must be done?

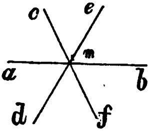
P.—A third line must be drawn so as to intersect *a b* and *c d;* and, then, we must ascertain whether the exterior angle is equal to the interior and opposite, or not; or, whether the alternate angles are co-equal or not; or, whether the two interior angles are, together, equal to, less than, or more than, two right angles.

M.—In the 6*th* case, what will happen if the non-parallel lines *a b*, *c d*, *e f*, be produced far enough?

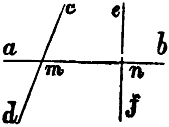
P.—They will meet and intersect each other.

M.—In how many points can they intersect?

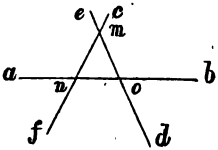
P.—In *one* point;



or, in *two* points;

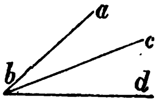


or, in *three* points.

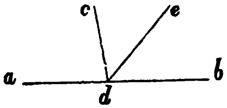


M.—If three lines *meet* in *one* point, how many angles can they make?

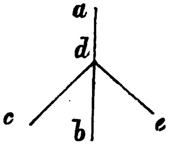
P.—Either two angles;

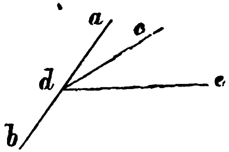


or, three angles;



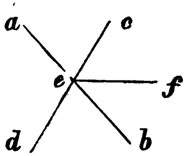
or, four angles;

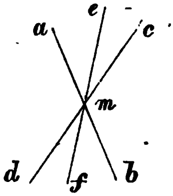




or, five angles;

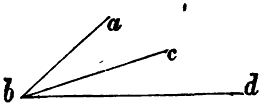
or, *six* angles.

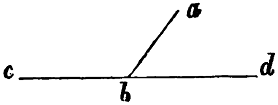


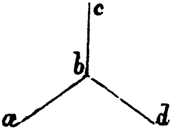


M.—If three lines, meeting in one point make *two* angles, what may these angles be ?

P.—Only acute angles: for, if the angles *a b c* and *c b d* were right angles the lines *a b* and *b d* would be in one straight line thus:—and, if the angles *a b c* and *c b d* were obtuse, the three lines *a b*, *c b*, and *d b* would make three angles, thus:—





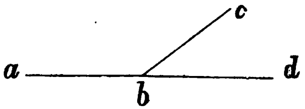


M.—Hence, if three lines, meeting in one point, make two right angles, what must be concluded?

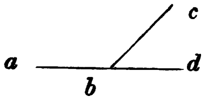
P.—That two of these lines are one and the same line.

M.—But, if three lines, meeting in one point, make two angles which, *together*, are equal to two right angles, what must then be concluded?

P.—That the lines *a b* and *d b* are but one and the same line; for, the line *c b* may be at right angles or not.



M.—Hence, if you were required to ascertain, whether *a b* and *d b* are one and the same line or not, what would you do?

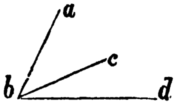


P.—We would endeavour to ascertain whether the angles, $abc + dbc = 2$ rt. $\angle$s, or not.

M.—If three lines, meeting in one point, make two angles, what may these angles be found as to equality when compared?

P.—Either equal or unequal.

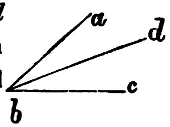
M—If the angles *a b c* and *c b d* are equal, what happens when the angles are placed upon each other, so that their *vertices* coincide?



P.—*a b* will fall upon *b d* and coincide with it.

M.—And, if $ab = bd$, what, then, will result?

P.—The point *a* will be upon the point *d.*



M.—But, if the angles *a b c* and *c b d* be unequal, and be so placed upon each other that their vertices coincide, will *a b* then be *upon b d?*

P.—No: *a b* will either be *within* the angle *c b d*, or, *without* it,—according as the angle *a b c* is *greater* or *less* than the angle *c b d.*

M.—What may the sum of these two angles be?

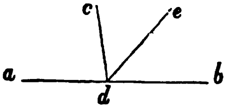
P.—They may, *together*, be either less than one right angle; or, they may be, together, equal to one right angle; or, they may be greater than *one* right angle, but they must be *less* than two right angles.

M.—If three lines make three angles, what is the sum of these angles?

P.—They are together equal to two or to four right angles.

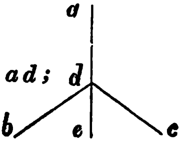
M.—Show this.

P.—



The 3 $\angle$s, $a d c + c d e + e d b = 2 \angle$s, $a d c + c d b$.
But the 2 $\angle$s, $a d c + c d b = 2$ rt. $\angle$s;
therefore, the 3 $\angle$s, $a d c + c d e + e d b =$ likewise, 2 rt. $\angle$s.

Next, produce any of the three lines, *a d;*



the 3 $\angle$s $a d b + a d c + b d c =$ the 4 $\angle$s $a d b +$ $+ a d c + b d e + e d c$.

But the 4 $\angle$s, $adb + adc + bde + edc = 4$ rt. $\angle$s; therefore, the 3 $\angle$s, $adb + adc + bdc = 4$ rt. $\angle$s.

M.—If three lines, meeting in one point, make either four, five, or six angles, what is their sum in each case?

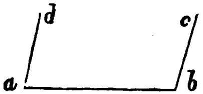
P.—Four right angles.

M.—What, then, is the sum of all the angles about one point?

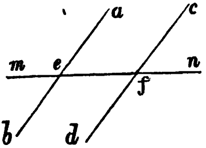
P.—Always four right angles.

M.—If three lines meet and intersect each other in two points, what is the least, and what the greatest, number of angles which they can form?

P.—The least number is *two* angles;



and the greatest number, *eight* angles.



M.—In the former case, if each of the angles, *d a b* and *c b a*, is a right angle, what must we conclude?

P.—That the lines *d a* and *c b* are *parallel.*

M.—And if their sum is = 2 rt. $\angle$s?

P.—That the lines *d a* and *c b* are parallel.

M.—But, if their sum be less or greater than 2 rt. $\angle$s?

P.—If their sum be less, *b a* and *c d* must meet toward the points *c* and *d;* and if their sum be greater than 2 rt. $\angle$s, the lines *d a* and *c b* will not meet.

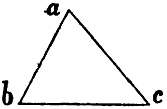
M.—In the former case, the lines *d a* and *c b* are said to *converge* towards *c* and *d*; and, in the latter case, they are said to *diverge*.

M.—If three lines, intersecting each other in two points, make *eight* angles, what is their sum?

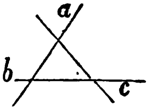
P.—Their sum is equal to 8 rt. ∠s.

M.—If three lines meet and intersect each other in three points, what is the least, and what is the greatest, number of angles they make?

P.—The least number is *three* angles;



and the greatest number is *twelve* angles.



M.—What do you observe respecting these lines?

P.—They inclose a space—they form a *triangle*.

SUBSTANCE OF THIS SECTION.

18. If two parallel lines intersect another straight line, they make the *exterior* angle equal to the *interior* and opposite angle.

19. If two parallel lines intersect another straight line, they make the *alternate* angles equal to each other.

20. If two parallel lines meet another straight line, they make the two interior angles together equal to two *right* angles.

21. If two straight lines intersect another straight line, so as to make the exterior angle not equal to the interior and opposite angle, these two lines are *not* parallel.

22. If two straight lines intersect another straight line, so as to make the alternate angles *not* equal to each other, these two lines are *not* parallel.

23. If two straight lines meet another straight line, and make the two interior angles, together, less than two right angles, these lines will meet if produced far enough on that side of the line on which are these interior angles.

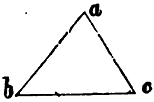
24. If, at one point in a straight line, two other straight lines make the adjacent angles, together, equal to two right angles, these two lines are in the *same* straight line.

SECTION III.

ONE TRIANGLE.

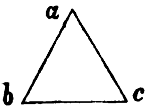
M.—State all you have learnt concerning triangles. (See Lesson IV.)—We shall now examine triangles more minutely. Can there be a triangle of which all the angles are right angles?

P.—No: for, if each of the angles *a b c* and *a c b* were a right angle, the lines *a b* and *a c* would be parallel and, therefore, could not meet at the point *a*; nor could *a b* and *b c* meet if the ∠s, *b a c* and *b c a* were right angles,—nor *a c* and *b c* if the ∠s, *c a b* and *c b a* were right angles.



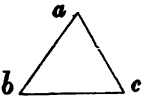
M.—Try if two of the angles can be right angles.

P.—They cannot; there cannot be a triangle of which two angles are right angles: for, if ∠s, *a b c* and *a c b* were right angles, *a b* and *a c* would be parallel, and, therefore, could *not meet* at the point *a*.



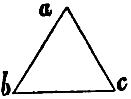
M.—What, then, must be concluded with respect to the sum of any two angles of every triangle?

P.—The sum of any two angles of every triangle must be less than 2 rt. $\angle$s: for, if $\angle$s, $a\,b\,c + a\,c\,b = 2$ rt. $\angle$s, the lines $a\,b$ and $b\,c$ would be parallel, and could not, therefore, meet at a.



M.—Can one of the angles of a triangle be a right angle? Such a triangle is called a *right-angled* triangle. Can there be a triangle of which the three angles are obtuse angles; or, of which two angles are obtuse angles; or, of which one angle is obtuse?

P.—The three angles of a triangle cannot be obtuse; because, if $\angle$s $a\,b\,c$ and $a\,c\,b$ are obtuse, then $\angle$s, $a\,b\,c + a\,c\,b$ are greater than 2 rt. $\angle$s; and, therefore, $a\,b$ and $a\,c$ cannot meet at a, but must diverge continually: and the same may be said of the lines $a\,b$ and $b\,c$, or $a\,c$ and $c\,b$.



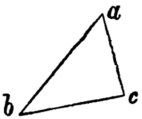
For the same reason, there cannot be a triangle of which two angles are obtuse: but the one angle of a triangle may be *obtuse.*

M.—Such a triangle is called an *obtuse-angled* triangle.—Can the three angles of a triangle be *acute* angles?

P.—The three angles of a triangle may be acute.

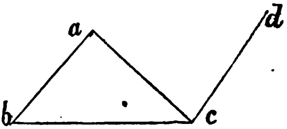
M.—What may be concluded from what you have discovered concerning the sum of the angles of every triangle?

P.—The sum of the angles of every triangle cannot be three right angles; neither can it be three obtuse, that is, more than three right angles: it must therefore, be less than three right angles.



M.—Let *a b c* be a triangle: find the sum of the angles *a b c*, *a c b*, and *b a c*.

It is advisable, here and in similar cases, to leave the pupils, unassisted, to the operation of their own faculties; and, should their efforts not prove successful, the time, thus occupied, must not be considered mis-spent,—inasmuch as such efforts will tend, perhaps more than anything else, to impart to them that habit of independent and patient thought and research, which constitutes a fundamental element of the mathematical character, as well as of every well-trained mind. All answers to such questions as the last should be submitted to the criticism of the *class*:—and the master may, ultimately, assist their endeavours, or correct and arrange their results—after the following example:—



M.—In the triangle *a b c*, draw from the point *c* a line, *c d*, parallel to one of the sides of the triangle. Which side must that be?

P.—Parallel to the side *a b*.

M.—What have we, thus, obtained?

P.—The angle *a c d*.

M.—And, to what angle is *a c d* equal?

P; $\angle a c d = \angle b a c$, for, they are alternate angles, formed by the parallel lines $a b$, $c d$, meeting the straight line $a c$.

M.—The three angles of the triangle are, therefore, together, as much as—what other angles?

P.—The angles of the triangle,

$a b c + a c b + b a c = \angle$s, $a b c + a c b + a c d$.

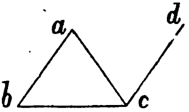
M.—And to what are the $\angle$s, $a b c + a c b + a c d$ equal?

P.—$\angle$s, $a b c + a c b + a c d = 2$ rt. $\angle$s; because they are equal to $\angle$s, $a b c + d c b$, which are equal to 2 rt. $\angle$s,—for, they are the two interior angles formed by the parallel lines $a b$ and $d c$ meeting the line $b c$.

M.—What must thence be concluded?

P.—That the angles, $a b c + a c b + b a c$, of the triangle $a b c$ are, also, equal to two right angles.

M.—State, now, in *writing*, what we have just said.



P.—From the point c of the triangle $a b c$ draw the line $c d$ parallel to $a b$;

then, because $a b$ and $c d$ are parallel,
the $\angle\, b a c = \angle$ altern. $a c d$,
To each of these angles add $\angle$s, $a c b + a b c$;
then, $\angle$s, $b a c + a c b + a b c = \angle$s, $a c d + a c b + a b c$.
But, $\angle$s, $a c d + a c b + a b c = 2$ rt. $\angle$s.
Therefore, $\angle$s, $b a c + a c b + a b c$, likewise, $= 2$ rt. $\angle$s.

M.—What axioms did you use in this demonstration?

P.—If equals be added to equals, the sums are equal; and, things which are equal to the same thing are equal to each other.

M.—What has been done, in order to demonstrate this truth—"the sum of the angles of a triangle is equal to two right angles?"

P.—A line has been drawn, from the point *c*, parallel to one of the sides of the triangle.

M.—Such a line is called an *auxiliary line;* for, it is by the *help* of this line that we have been enabled to demonstrate this truth. Is it necessary to draw an auxiliary line from the point *c exclusively?*

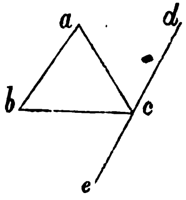
P.—No: a line may be drawn from the point *a*, parallel to the side *b c*, or from the point *b*, parallel to the side *a c*.

M.—In general, then, an auxiliary line may be drawn from *either* of the points *a*, *b*, *c*, parallel to what side?

P.—That side which is opposite to the angle from the vertex of which the parallel line is drawn.

M.—Produce the auxiliary line *c d* from the point *c;* what do you obtain?

P.—The angle *b c e.*



M.—Endeavour to demonstrate the same truth by means of the two angles $a\ c\ d$, $b\ c\ e$.

P. $\angle\ a\ c\ d =$ alternate $\angle\ b\ a\ c$,
and $\angle\ b\ c\ e =$ alternate $\angle\ a\ b\ c$;
therefore, $\angle$s, $a\ c\ d + b\ c\ e = \angle$s, $b\ a\ c + a\ b\ c$.
To each of these equals add $\angle\ b\ c\ a$;
then, $\angle$s, $a\ c\ d + b\ c\ e + b\ c\ a = \angle$s, $b\ a\ c + a\ b\ c + b\ c\ a$:
But $\angle$s, $a\ c\ d + b\ c\ e + b\ c\ a = 2$ rt. $\angle$s;
therefore, $\angle$s, $b\ a\ c + a\ b\ c + b\ c\ a$, likewise, $= 2$ rt. $\angle$s.

M.—Find whether or not the sum of the angles of *every* triangle is equal to two right angles. What, then, must each of the angles of an equi-angular triangle be?

P.—The third part of two right angles; that is, two thirds of one right angle.

M.—If one of the angles of a triangle be a right angle, what will be the sum of the other two angles?

P.—The sum of the other two angles must be equal to what remains after the subtraction of one right angle from two right angles,—that is, to one right angle.

M.—Hence, what are the angles of a right-angled triangle?

P.—One angle is a right angle, and each of the other two angles is acute.

M.—If one of the angles of a triangle is obtuse, what is the sum of the other two angles, and what is each of them?

P.—The sum of the other two angles must be less than one right angle; and, therefore, each of them must be *acute*.

M.—If one of the angles of a triangle be acute, what must be the sum of the other two angles, and what may each of them be?

P.—Their sum must be greater than one right angle: one of them may, therefore, be a right, the other an acute angle; or, one of them may be obtuse, and the other acute; or, each of them may be acute.

M.—If it be known that the sum of two angles of a certain triangle is equal to one right angle, what may be concluded with respect to the triangle?

P.—As the remaining must be a right angle, the triangle must be a right-angled triangle.

M.—If it be known that the sum of two angles of a certain triangle is less than one right angle, what may be concluded with respect to the triangle?

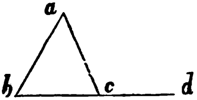
P.—The remaining angle being obtuse, the triangle must be obtuse-angled.

M.—And, if it be known that the sum of two angles of a certain triangle is greater than one right angle, what may be concluded?

P.—The remaining angle must be acute; but, the triangle may be right-angled, obtuse-angled, or acute-angled.

M.—What may we obtain by producing one of the sides of a triangle?

P.—An angle such as *a c d.*



M.—How is this angle situated with respect to the triangle?

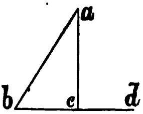
P.—It is *without* the triangle.

M.—On that account it is called an *exterior* angle. By what name would you designate the angle *a c b*?

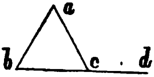
P.—I would say that the angle *a c b* is *adjacent* to the exterior angle *a c d*.

M.—The other two angles *b a c* and *a b c* are called the *interior* and *opposite* angles with respect to the exterior angle *a c d*. Now compare the exterior angle *a c d* with each of the interior angles of the triangle *a b c*, and state the various results of the comparison.

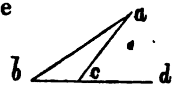
P.—1. The exterior angle *a c d* may be *equal* to its adjacent angle *a c b*; in this case each of them is a right angle, and the triangle *a b c* is, therefore, a right angled-triangle.



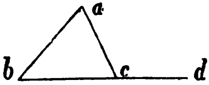
2. The exterior angle *a c d* may be *greater* than its adjacent angle *a c b*: the exterior ∠ *a c d* is, then, obtuse, the adjacent ∠ *a c b* being, therefore, acute; and the triangle may be a right-angled, obtuse-angled, or acute-angled triangle,—because either of the interior opposite angles *a b c*, *b a c* may be right, obtuse, or acute (page 64).



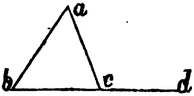
3. The exterior angle *a c d* may be *less* than its adjacent angle *a c b*. The exterior ∠ *a c d* is, then, acute, the adjacent ∠ *a c b* obtuse, and the triangle *a b c*. therefore, obtuse-angled.



4. The exterior angle *a c d* is always *greater* than the interior



and opposite angle abc: for,—
because $\angle$s, $acd + acb = 2$ rt. $\angle$s,
and $\angle$s, $abc + acb$ are less than 2 rt. $\angle$s;
therefore, $\angle$s, $acd + acb$ are greater than $\angle$s, $abc + acb$:
from each of these unequals if you take away $\angle\, acb$,
there remains the exterior $\angle\, acd$, *greater* than the interior and opposite angle abc.



5. The exterior angle acd is, likewise, greater than the other interior and opposite angle bac:
for, because $\angle$s, $acd + acb = 2$ rt. $\angle$s,
and $\angle$s, $bac + acb$ are less than 2 rt. $\angle$s;
therefore, $\angle$s, $acd + acb$ are greater than $\angle$s, $bac + acb$:
from each of these unequals if you take away $\angle\, acd$,
the exterior $\angle\, acd$ remains *greater* than the interior and opposite $\angle\, bac$.

M.—Use the sign $<$ for the words "is or are less than,"
and the sign $>$ for the words "is or are greater than;"
also, the sign $\therefore$ for the word "therefore,"
and the sign $\because$ for the word "because."
Express Nos. 4 and 5 in words.

P.—If one side of a triangle be produced, the exterior angle is greater than either of the interior and opposite angles.

M.—Write the demonstration of this truth on your slate, and introduce the *signs* just recommended.

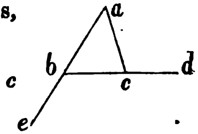
P. $\because \angle\text{s}, acd + acb = 2 \text{ rt. } \angle\text{s},$
and $\angle\text{s}, abc + acb < 2 \text{ rt. } \angle\text{s};$
$\therefore \angle\text{s}, acd + acb > \angle\text{s}, abc + acb.$
From each of these unequals take away $\angle\, acb$—
Then, the remaining ext. $\angle\, acd >$ the int. opp. $\angle\, abc.$

In the same way, it may be shown that the ext. $\angle\, acd$ is greater than the int. opp. $\angle\, bac.$

M.—Compare the exterior angle of a triangle with the sum of the two interior opposite angles.

P.—The exterior angle a c d is equal to the sum of the two interior opposite angles a b c + b a c: for,
$\because \angle\text{s}, acd + acb = 2 \text{ rt. } \angle\text{s},$
and, also, $\angle\text{s}, acb + abc + bac = 2 \text{ rt. } \angle\text{s};$
$\therefore \angle\text{s}, acd + acb = \angle\text{s}, acb + abc + bac.$
Take away $\angle\, acb$ from each of these equals—
$\therefore$ the remaining ext. $\angle\, acd =$ remaining int. opp. $\angle\text{s}, abc + bac.$

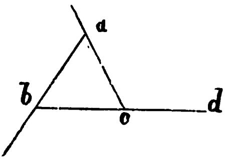
M.—Produce two of the sides of a triangle. What is the sum of the two exterior angles?



P. $\angle\text{s}, acd + acb = 2 \text{ rt. } \angle\text{s},$
and $\angle\text{s}, ebc + abc = 2 \text{ rt. } \angle\text{s};$
$\therefore \angle\text{s}, acd + acb + ebc + abc = 4 \text{ rt. } \angle\text{s};$
and, $\therefore \angle\text{s}, acd + acb + ebc + abc + bac > 4 \text{ rt. } \angle\text{s}.$
From these unequals take $\angle\text{s}, acb + abc + bac,$ which $= 2 \text{ rt. } \angle\text{s};$
Then, the remaining ext. $\angle\text{s}, acd + ebc > 2 \text{ rt. } \angle\text{s}.$

Hence, two exterior angles of a triangle are together *greater* than two right angles.

M.—Produce each of the sides of a triangle. What is the sum of the three exterior angles?



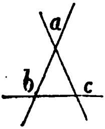
P.—Each exterior angle together with its adjacent angle = two rt. ∠s; ∴ the three ext. ∠s together with the angles of the triangle = six rt. ∠s.

But the ∠s of the triangle = 2 rt. ∠s;

∴ the three exterior ∠s = 6 rt. ∠s − 2 rt. ∠s = 4 rt. ∠s.

Hence, if each of the sides of a triangle be produced, the sum of the three exterior angles = four rt. ∠s.

M.—Produce each of the sides of a triangle both ways. What is the sum of the twelve angles; and, what is the sum of the nine exterior angles?



P.—The sum of the twelve angles = 12 rt. ∠s; and ∴ the sum of the 9 ext. ∠s = 10 rt. ∠s.

SUBSTANCE OF SECTION III.

1. Any two angles of a triangle are together less than two right angles.

2. The interior angles of every triangle are together equal to two right angles.

3. A right-angled triangle is that which has a *right* angle.

4. An obtuse-angled triangle is that which has an *obtuse* angle.

5. An acute-angled triangle is that which has *three acute* angles.

6. If one side of a triangle be produced, the exterior angle is greater than either of the interior and opposite angles.

7. If one side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles.

8. If each of the sides of a triangle be produced, the three exterior angles are, together, equal to four right angles.

SECTION IV.

TWO TRIANGLES—THEIR EQUALITY.

M.—What may be said, on comparing the angles of two triangles?

P.—1. The angles of one triangle are, together, equal to the angles of any other triangle; because, their sum, in each, is equal to two right angles.

2. One angle of the one may be equal to an angle of the other.

3. Two angles in the one may be equal to two angles in the other, each to each.

4. The three angles of the one may be *equal* to the three angles of the other, each to each.

5. The three angles of the one may be *un*equal to the three angles of the other, each to each.

M.—If an angle of one triangle be equal to an angle of another triangle, what may be said of the other two angles, in each?

P.—The sum of the other two angles of the one triangle must be equal to the sum of the remaining two angles of the other; because, the sum of the three angles of the one is equal to the sum of the three angles of the other, and if the equal angles be subtracted from these equals, the remaining angles must be equal.

M.—If two triangles have two angles of the one equal to two angles of the other, each to each, what may be said of the remaining third angles?

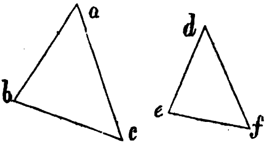
P.—They must be equal,—for the reason alleged in the former case.

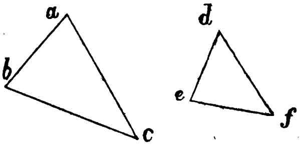
M.—What may be said, then, of the angles of these triangles?

P.—The angles of the one are equal to the angles of the other, each to each.

M.—Draw two triangles having one angle of the one equal to one angle of the other.

P.—Let the angle *b a c* be equal to the angle *e d f.*



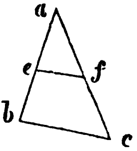


M.—If the angle *b a c* be equal to the angle *e d f,* what happens, if the triangle *e d f* be placed upon the triangle *a b c,* so, that the point *d* may be upon the point *a,* and the angle *e d f* upon the angle *b a c?*

P.—The side *e d* must fall upon the side *a b,* and the side *d f* upon the side *a c.*

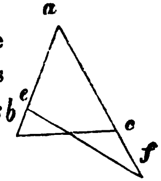
M.—And, where may the points *e* and *f* fall?

P.—Either somewhere on the sides *a b* and *a c*,—when *d e* and *d f* are, each, less than *a b* and *a c;*



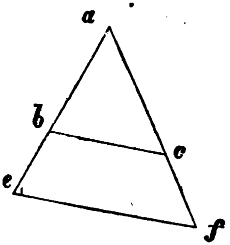
Or,

one of them may fall upon *a b* and the other beyond the point *c*,—when *d e* is less than *a b*, and *d f* is greater than *a c;*



Or,

both may fall beyond the side *b c*, without the triangle *a b c*, —when *d e* and *d f* are both greater than *a b* and *a c*.



M.—And, when will the points *e* and *f* fall exactly upon the points *b* and *c?*

P.—When $d\,e = a\,b$, and $d\,f = a\,c$.

M.—And where, then, will the side *e f* fall?

P.—*e f* must fall upon *b c*, and *e f* must, also, be equal to *b c;* because, since the point *e* falls upon *b*, and the point *f* upon *c*, the whole line *e f* must fall upon the whole line *b c*, and be equal to it,—for, *e f* and *b c* are *straight*, not curved, lines.

M.—And, what may be said of the remaining two angles of each triangle?

P.—The angle *d e f* must be equal to the angle *a b c*, and the angle *d f e* must be equal to the angle *a c b*.

M.—And what may, then, be said of the triangles themselves?

P.—The triangle *d e f* must be equal to the triangle *a b c*.

M.—Then, when are two triangles equal to each other?

This question the author has been accustomed to propose to his class previously to the preceding investigation, which is strictly in accordance with the demonstration of Euclid. (B. I. Prop. 4). The answers of his pupils were, however, generally of such a nature as to render the demonstration too loose and unmathematical; and he has, accordingly, found it necessary to lead their thoughts into a chain of reasoning similar to the preceding.

P.—Two triangles are equal, when two sides of the one, with the included angle, are equal to two sides of the other, each to each, with the included angle.

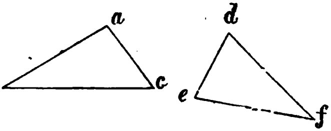
M.—Instead of, "the included angle," or "the angle between them," say, "the angle contained by them." What may be said of the third sides of two such triangles?

P.—Their third sides are equal.

M.—And, of the remaining angles in each?

P.—The remaining angles of the one are equal to the remaining angles of the other, each to each.

M.—Then, if $ab = df$, and $ac = de$, and $\angle bac = \angle edf$, state what you know of the triangles abc and def.



P.—The side $bc =$ side ef,
$\angle abc = \angle dfe$,
and $\angle acb = \angle def$.

M.—Could the angle abc be equal to the angle def?

P.—No: because if the triangle def be applied to the triangle abc so that the point a may be on the point d, and the side ab upon the side, equal to it, df, the side ac will fall upon de,—
because $\angle bac = \angle edf$;
the point f will fall upon the point b,—
because $ab = df$;
the point c will fall upon the point e,—
because $ac = de$;
and, hence, the third side bc must fall upon ef, and be, therefore, equal to ef.
Also the triangle abc must fall upon the triangle def, and be, therefore, equal to it; and the $\angle abc$ upon the $\angle dfe$, and $\angle acb$ upon $\angle def$.
Hence, these angles must be equal: and the angle abc cannot be equal to the angle def, unless $\angle def$ be likewise equal to $\angle dfe$.

M.—What may be said of the sides which are opposite to the equal angles, in each triangle?

P.—The sides opposite to the equal angles are equal.

M.—And what, of the angles to which the equal sides, in each triangle, are opposite?

P.—The angles to which the equal sides are opposite are equal.

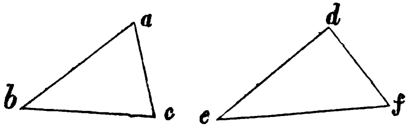
M.—Now, state connectedly the different truths we have established respecting two such triangles.

P.—If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other—*their third sides are equal—the triangles are equal*—and *their other angles are equal, each to each*, namely, those to which the equal sides are opposite.

M.—Demonstrate this truth on your *Slates.*

The pupils must give a demonstration in all respects similar to the preceding; and great attention should be paid to neatness of performance, correctness of statement, and methodical arrangement of the several parts.

M.—



But, if in the triangles *a b c* and *d e f*,

$a\,b = d\,e,\ a\,c = d\,f,$

and $\angle\, e\,d\,f$ is greater than $\angle\, b\,a\,c$,

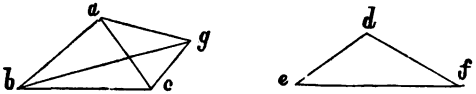
what will necessarily be concluded with respect to their third sides or *bases*, *e f* and *b c*?

P.—The base *e f* must be greater than the base *b c*.

M.—State this deduction at full length.

P.—If two triangles have two sides of the one equal to two sides of the other, each to each—but, the angle contained by the two sides of the one greater than the angle contained by the two sides, equal to them, of the other—the base of that triangle which has the greater angle, is greater than the base of the other.

N.B. This theorem may be demonstrated, by making

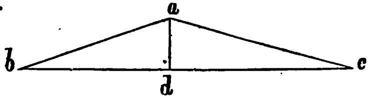


$\angle\, b\,a\,g = \angle\, e\,d\,f$, and $a\,g = d\,f$ or $a\,c$;
and by joining *c g* and *b g*—(Euclid, B. I. Prop. 24): though it may, perhaps, be desirable to defer the demonstration until the *rehearsal* of this section.

M.—State what you know of an *isosceles* triangle. (Introduct. Lesson IV.)

P.—An isosceles triangle is that of which two sides are equal: the third, the unequal, side is called the base; the angles adjacent to the base are called the angles *at* the base.

M.—Compare the angles at the base of an isosceles triangle.



P.—In the triangle *a b c*,
let $a\,b = a\,c$;
draw the straight line *a d*, bisecting $\angle\, b\,a\,c$:
then, $\because\ a\,b = a\,c$, and *a d* is common to the two triangles, *a d b* and *a d c*,

and the $\angle\, b\,a\,d = \angle\, c\,a\,d$:

$\therefore$ the third side $b\,d =$ third side $c\,d$,

the triangle $a\,d\,b =$ triangle $c\,a\,d$,

and the remaining angles of the one are equal to the remaining angles of the other, each to each, namely, those to which the equal sides are opposite;

and $\therefore\ \angle\, a\,d\,b = \angle\, a\,d\,c$,

and $\angle\, a\,b\,d = \angle\, a\,c\,d$—the angles *at* the base.

Then, *in an isosceles triangle, the angles at the base are equal to each other.*

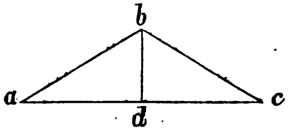
(Should the pupils be unable to discover the manner of drawing the auxiliary line $a\,d$, the master may thus far assist them, then leaving them to find out the demonstration, by themselves. The demonstration of Euclid (B. I. Prop. 5.) is, at this stage, too *prolix*, and may be dispensed-with till they are properly prepared to enter on the study of his Elements.

M.—From the manner in which $a\,d$ has been drawn, what other properties can you discover in an isosceles triangle?

P.—When the angle opposite to the base of an isosceles triangle is bisected,

1. The base is, likewise, bisected;
2. The bisecting line is at right angles to the base.

M.—The angle opposite to the base of an isosceles triangle is usually called the *angle at the vertex.* In an isosceles triangle, if the base be bisected, and, from the point of bisection, a straight line be drawn to the angle at the vertex, how will such a line *affect* this angle?



P.—The angle abc at the vertex is bisected by bd;
for $\because$ $ab = bc$, and
$ad = cd$ (ac being bisected),
and $\angle bac = \angle bcd$;
$\therefore$ the triangles abd and cbd are equal in every respect.
Hence $\angle abd = \angle cbd$;
$\therefore$ $\angle abc$, at the vertex, is bisected.

M.—How are the *equal* angles of an isosceles triangle situated with regard to the *sides*?

P.—The two equal angles are opposite to the two equal sides.

M.—Hence, if two angles of a triangle be equal to each other, what must follow?

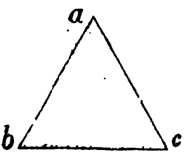
[In an introductory course of study, it is advisable to omit demonstrations of the *converse* truths; and, accordingly, few will be found in this treatise. Indeed, the *indirect* method of demonstration (reductio ad absurdum),—as, among other reasons, involving the assumption of an *un*truth, and being the least satisfactory, — is unsuitable to beginners or *young* learners; and it should, therefore, be avoided, as much as possible, in initiatory instruction.]

P.—The *sides* which are opposite to these two equal angles are equal to each other—that the triangle is *isosceles.*

M.—Hence, if the three sides of a triangle are equal, — that is, if a triangle is *equilateral,* — what may be said of the angles?

P.—The angles of an equilateral triangle are equal: an equilateral triangle is *equi-angular.*

M.—Demonstrate this.



P.—Let $a\,b\,c$ be an *equilateral* triangle;
then $\because a\,b = a\,c$,
$\angle a\,b\,c = a\,c\,b$;
and $\because a\,b = b\,c$,
$\angle b\,a\,c = a\,c\,b$:
$\therefore \angle a\,b\,c = \angle b\,a\,c = \angle a\,c\,b$;
and $\therefore$ the triangle $a\,b\,c$ is *equiangular.*

M.—And, if a triangle be equiangular, what may be said of its sides?

P.—If a triangle is equiangular, it is also equilateral.

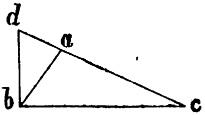
M.—Since a triangle of which the sides are equal has likewise its angles equal, what may be inferred if the sides be *un*equal?

P.—That the angles are, likewise, unequal.

M.—And, what angle must, then, be greater than either of the others?

P.—That to which the greater side is opposite.

M.—Demonstrate this, by means of those truths with which you are already acquainted.



P.—In the triangle $a\,b\,c$, let $b\,c$ be greater than $a\,c$; the angle $b\,a\,c$, opposite $b\,c$, shall be greater than the angle $a\,b\,c$, opposite $a\,c$:
$\because a\,c$ is less than $b\,c$.
Produce $b\,c$ at a, and make $c\,d = b\,c$,
and join $d\,b$;
then $\because b\,c = d\,c$,

$\angle c b d = \angle c d b$;
$\therefore \angle c d b > \angle a b c$.
But $\angle b a c >$ int. opp. $\angle c d b$;
much more $\therefore \angle b a c > \angle a b c$.

Hence, *in every triangle the greater angle is opposite to the greater side.*

M.—If, then, it be known that one angle of a triangle is greater than another angle, what must be concluded with respect to the sides subtending [opposite to] these angles?

P.—*The side subtending the greater angle must be greater than the side subtending the less.*

M.—Which, then, of the sides of a right-angled triangle is the greatest?

P.—The side subtending the right angle.

M.—And in an obtuse-angled triangle, which is the greatest side?

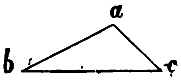
P.—The side subtending the obtuse angle.

M.—In an equilateral triangle, compare the sum of any two sides with the remaining side.

P.—Any two sides of an equilateral triangle must together be greater than the remaining [third] side, because all the sides are equal.

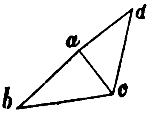
M.—But, in any triangle, are two sides together greater or less than the remaining third side?

P.—The sides, $b a + a c$ must be greater than $b c$, because $b c$ is the shortest distance between the points b and c.

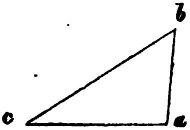


For same reason, $a b + b c > a c$, and $a c + b c > a b$.

M.—This truth may be demonstrated by converting two sides into one: endeavour to do so.



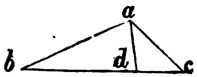
P.—Produce the side *a b* at the point *a*, and make *a d*, the part produced, equal to *a c*, and join *d c*:
then ∵ $ad = ac$,
$\angle adc = \angle acd$;
∴ $\angle bcd > \angle adc$.
And ∵ to the greater angle the greater side is opposite;
∴ $bd > dc$,
but $bd = ba + ac$:
∴ $ba + ac > bc$.
Hence, *any two sides of a triangle are together greater than the remaining side.*



M.—In the triangle *a b c*, let the side *b c* be greater than *a b*; how would you discover the excess of *b c* over *a b*—*i. e.* by how much *b c* is greater than *a b*?

P.—By cutting off from the greater, *b c*, a part equal to *a b*; the remaining part must be the difference, in length, between *b c* and *a b*.

M.—Compare this difference between two unequal sides of a triangle with the remaining [third] side.

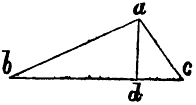


P.—Let *b c* be greater than *a b*;
from *b c* cut off $bd = ab$:
d c is the difference between *b c* and *a b*.

Then, because any two sides of a triangle are together greater than the third side,
$ba + ac > bc$;
but $ba = bd$;

$\therefore bd + ac > bc.$
From these unequals take away bd, which is common to both—
there remains $ac > dc$.
Hence, *the difference between any two sides of a triangle is less than the third side.*

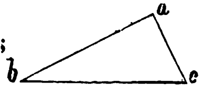
M.—The same truth can be demonstrated by means of the angles: try this method.



P.—Let dc be the difference between bc and ab; join ad:
then $\because ba = bd$,
$\angle bad = \angle bda$;
but the ext. $\angle adc >$ int. opp. $\angle bad$;
$\therefore \angle adc > \angle bda$.
Also, the ext. $\angle bda >$ int. opp. $\angle dac$;
much more $\therefore \angle adc > \angle dac$;
but to the greater angle the greater side is opposite —
$\therefore ac > dc$—
that is, the difference, dc, between bc and ab, is *less* than the third side, ac.

M.—Compare the three sides of a triangle with the double of any one side.

P.—The three sides of any triangle are together greater than double the length of any one side;
for, $ab + ac$ being $> bc$,
add bc to each of these unequals;
then $ab + ac + bc > bc + bc$.



SUBSTANCE OF SECTION IV.

1. If two triangles have one angle of the one equal to one angle of the other, the sum of the remaining

two angles of the one is equal to the sum of the remaining two angles of the other.

2. If two triangles have two angles of the one equal to two angles of the other, each to each, the third angle of the one is equal to the third angle of the other; that is, the triangles are equiangular.

3. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by these sides equal, their third sides are equal, the triangles are equal, and the remaining angles of the one are equal to the remaining angles of the other, each to each, namely, those to which the equal sides are opposite.

4. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides, equal to them, of the other, the base of that triangle which has the greater angle is greater than the base of the other.

5. The angles at the base of an *isosceles* triangle are equal.

6. In an isosceles triangle, the straight line which bisects the vertical angle bisects the base.

7. In an isosceles triangle, the straight line which bisects the vertical angle stands at right angles to [is *perpendicular* to,] the base.

8. In an isosceles triangle, if the base be bisected, the straight line joining the vertical angle and the point of bisection bisects the vertical angle and stands at right angles [is *perpendicular* to,] to the base.

9. If a triangle be equilateral, it is likewise equiangular.

10. If two angles of a triangle be equal to one another, the sides opposite to them are likewise equal; that is, the triangle is isosceles.

11. If a triangle be equiangular, it is likewise equilateral.

12. In every triangle the greater side is opposite to the greater angle.

13. In every triangle the greater angle is subtended by the greater side.

14. Any two sides of a triangle are together greater than the third side.

15. The difference between any two sides of a triangle is less than the third side.

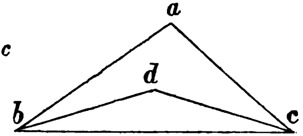
16. The three sides of every triangle are together greater than double the length of any one side.

SECTION IV.

EQUALITY OF TRIANGLES.

M.—Describe a triangle; in it take any point, and, from the extremities of any one side of the triangle, draw lines to that point: what is the result?

P.—Two triangles, *a b c* and *b d c*.



M.—What have these two triangles *in common?*

P.—The base *b c*.

M.—Compare the sum of the sides *b d* and *d c* with the sum of the sides *b a* and *a c.*

Obs.—As it is important that the pupils should find-out a method of demonstration for themselves, the master ought, in this and every similar instance, to withhold assistance as long as he perceives the majority of the class actively engaged in the investigation of the question. If, ultimately, the pupils should not succeed in discovering a demonstration, he may direct their attention to the main points in the question: thus, with respect to the preceding—

M.—What are you required to do?

P.—To compare $b\,d + d\,c$ with $b\,a + a\,c$.

M.—What does that mean?

P.—To try whether $b\,d + d\,c$ is equal to, or greater, or less than $b\,a + a\,c$.

M.—When the sides of triangles are to be compared with each other, which of the preceding truths will guide you?

P.—"The greater side subtends the greater angle;" or, "any two sides of a triangle are together greater than the third."

M.—If you adopt the former of these truths, how must you draw a line so as to find a relation between *b a* and *b d*?

P.—We must join the points *a* and *d.*

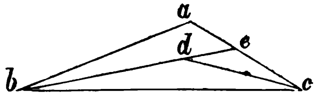
M.—Do this, and see if it will assist you.

The pupils will find that it cannot assist them, because the point *d* is not determined.

M.—And, if you wish to use the other truth, you have mentioned, what must be done?

Here, the master should leave the pupils to their own resources: they will, doubtless, find that either *b d* or *c d* must be produced,—if the preceding lessons have been thoroughly understood.

When any of the pupils have succeeded, let the master describe a triangle on the *large* school-slate, and the successful pupil submit his demonstration to the class, the master writing it down as the pupil proceeds. Thus:



(*Pupil dictating, and the master writing.*)

Produce bd to e;

then, $ba + ae$ being $> be$,

add ec to each of these unequals—

$\therefore ba + ac > be + ec$.

Also, $de + ec$ being $> dc$,

add bd to each of these unequals—

$\therefore be + ec > bd + dc$.

But, it has been shown that

$ba + ac > be + ec$;

much more $\therefore$ is $ba + ac > bd + ec$.

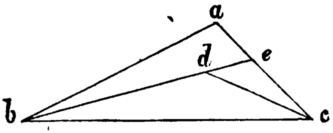
Hence, if a point be taken in a triangle, the straight lines drawn to it from the extremities of any one side are, together, less than the other two sides of the triangle.

The master may, now, let the rest of the class read [not aloud] what is written on the slate; and, there-after, the demonstration being rubbed-out, let each pupil, in turn, be called-upon to give an *oral* demonstration of the problem.

If the class consists of many pupils, the recapitulation of the solution by *each* pupil is apt to occasion inattention, on the part of those who fully know it. It is, then, advisable to let each pupil take a share of the demonstration—not in regular rotation, but at the call of the master, in order, the more effectually, to sustain universal *attention*.

This exercise having been gone-through, let the pupils be called-upon to re-produce the whole in writing, on their own slates — the master carefully revising what they have written.

Should there be, after all, some, weaker, pupils incapable of doing so, the master may assist them by writing, on the school-slate, the main points of the problem, thus:



Show that

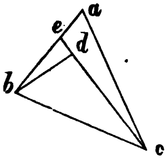
1. $ba + ac > be + ec$.

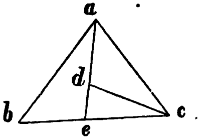
2. $be + ec > bd + dc$.

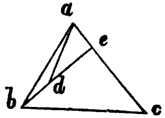
3. Draw the necessary inference.

The preceding process has been found serviceable even to those who fully understand the demonstration; as, they are thereby led to resolve the problem into its component parts.

Moreover, it is not an unnecessary practice to invert *diagrams* in all possible ways,—and, in this instance, to produce, now be, and, at other times, dc.

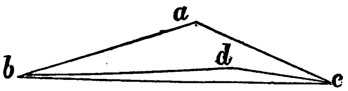


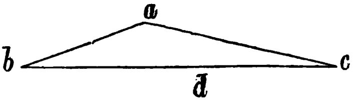




M. (*continues*).—Where must the point d be taken so that $bd + dc$ shall be less than any other two lines drawn to another point in the triangle?

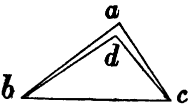
P.—The point d must be taken very near the side bc,—it must be taken in the side bc; thus:





M.—And where must the point d be taken so that $bd + dc$ may be greater than any other two lines drawn to another point in the triangle?

P.—The point d must be taken very near the vertex, a, of the triangle, thus:



M.—And, what is to be observed, if the point d is taken *in* the vertex a?

P.—The lines bd and cd, then, coincide with the sides, ba and ca, of the triangle.

M.—And, what may, then, be said of the two triangles abc and bdc?

P.—That the sides of the triangle bdc coincide with the sides of the triangle abc, each with each; and consequently, that the triangles are equal to each other.

M.—You are already acquainted with one instance of the *equality* of triangles; what is it?

P.—Two triangles are equal when they have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal.

M.—Now, I think, from what you have before remarked respecting the triangles abc and bdc, you will be able to find out another instance of the equality of triangles. What is it?

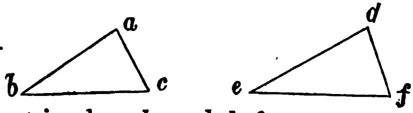
P.—Two triangles are equal when the three sides of the one are equal to the three sides of the other, each to each.

M.—Right; and what angles are equal in two such triangles?

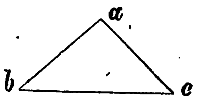
P.—Those angles which are opposite to the equal sides.

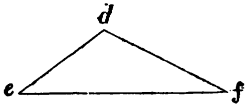
M.—Illustrate what you have said, by drawing two triangles having these requisites.

P.—



In the triangles abc and def,
if $ab = de$, $ac = df$, and $bc = ef$;
then, $\angle acb = \angle dfe$, $\angle abc = \angle def$, and
$\angle bac = \angle edf$,—
and $\triangle abc = \triangle def$.





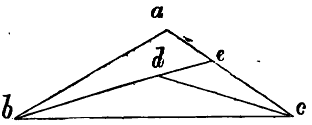
M.—If, then, in the triangles *a b c* and *d e f*,
$ab = de$ and $ac = df$,
but the base *e f* is greater than the base *b c*;
what must be concluded with respect to the angle *e d f*, which is opposite to the greater base?

P.—The angle *e d f* must be greater than the angle *b a c*.

M.—State this proposition fully, in words.

P.—If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that triangle which has the greater base is greater than the angle contained by the sides, equal to them, of the other.

M.—There is, yet, something to be remarked with regard to the angle which the lines *b d*, *c d* contain—(*describing the figure on the slate*).



Compare the angle *b d c* with the angle *b a c*.

P.—The angle *b d c* is greater than the angle *b a c*;
for ∵ exterior $\angle\, bec$ of $\triangle\, bae >$ int. opp. $\angle\, bac$,
and the exterior $\angle\, bdc$ of $\triangle\, dec >$ int. opp. $\angle\, bec$;
much more ∴ is $\angle\, bdc > \angle\, bae$.

The pupils repeat the demonstration and, then, write it on their own slates; the master, as before, writes upon the school-slate the following:

Show that

1. $\angle bec > \angle bac$.
2. $\angle bdc > \angle bec$.
3. Thence draw the necessary consequence.

M.—Hence, the angle formed by two lines drawn from the extremity of any side of a triangle to a point *within it* is greater than ——

P.—The angle contained by the other two sides of the triangle.

M.—Compare the angles *d b c* and *d c b* with the angles *a b c* and *a c b*.

P.—The angles *d b c* and *d c b* are, evidently, less than the angles *a b c* and *a c b*,—because they are only *parts* of the latter angles.

M.—Where must the point *d* be taken, so that the angles *d b c* and *d c b* may become equal to the angles *a b c* and *a c b?*

P.—The point *d* must be taken in coincidence with the point *a*.

M.—And what may, then, be said of the triangles *a b c* and *d b c?*

P.—They are equal to each other.

M.—After supposing, then, the angles *d b c* and *d c b* equal to the angles *a b c* and *a c b*, each to each, there are two *other* parts in the triangles *a b c* and *d b c*, which are equal to each other, or which these triangles have in common, if we consider them separated from each other. What are they?

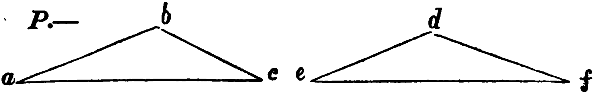
P.—The side bc.

M.—And how is this side situated with respect to the angles?

P.—It is adjacent to them.

M.—Describe two triangles having the following requisites: two angles, and the side adjacent to them, of the one, equal to two angles, and the side adjacent to them, of the other.

P.—



Let $\angle bac = \angle def$,
$\angle acb = \angle dfe$,
and $ac = ef$.

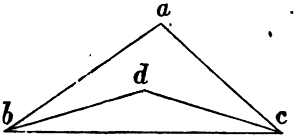
M.—If one of these triangles, we suppose to be applied to the other triangle, so that the point e may be upon the point a, and the side ef upon the side ac, what must happen?

P.—The point f must fall upon the point c,
because $ef = ac$;
and df must coincide with bc,
because $\angle dfe = \angle acb$;
and ed must coincide with ab,
because $\angle def = \angle bac$;
and, therefore, the point d must fall upon the point b,—and the triangle def must coincide with the triangle abc, and be equal to it.

M.—Here, then, is a third instance of equality in triangles: what is it?

P.—Two triangles are equal, when they have two angles of the one equal to two angles of the other, each to each, and have likewise the sides adjacent to the equal angles equal to each other.

M.—Repeat, now, all you have learnt from the investigation of the triangles *a b c* and *b d c*.



The pupils repeating, the master writes their statements on the slate. Thus :—

1. *b d* + *d c* is less than *b a* + *a c*.

2. *b d* and *d c* are least, when the point *d* is taken in the *side* c.

3. *b d* and *d c* are equal to *b a* and *a c*, when the point *d* is taken in the *vertex a*.

4. Hence, two triangles are equal when three sides of the one are equal to three sides of the other, each to each.

5. Again: if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides, equal to them, of the other.

6. The angle *b d c* is greater than the angle *b a c*.

7. The angles *d b c* and *d c b* are less than the angles *a b c* and *a c b*.

8. When the angles *d b c* and *d c b* are equal to the

angles *a b c* and *a c b*, each to each, the point *d* coincides with the vertex *a*.

9. Hence, two triangles are equal when they have two angles of the one equal to two angles of the other, each to each, and have likewise the sides adjacent to the equal angles equal to each other.

M.—And, if the angles *b d c* and *d b c* be equal to the angles *b a c* and *a b c*, each to each, what must follow?

P.—The angle *d c b* must be equal to the angle *a c b*; and the triangle *b d c* must coincide with the triangle *a b c*, and be equal to it.

M.—Hence, if two triangles have two angles of the one equal to two angles of the other, each to each, and have likewise one side equal to one side, how must these sides be situated in order that the triangles may be equal?

P.—They must either be adjacent to the equal angles, or they must be opposite to the equal angles.

M.—This, then, is a *fourth* instance of equality in triangles: state it.

P.—Two triangles are equal when they have two angles of the one equal to two angles of the other, each to each, and have likewise one side equal to one side,—namely, those opposite to the equal angles.

This truth the master, now, writes on the slate, in addition to the others already there. The whole is, then, committed to memory by the pupils.

A recapitulation of this paragraph is not given,—its *result* having just been written on the school-slate.

SECTION VI.

QUADRILATERAL FIGURES.

M.—We have, now, become acquainted with the most general truths respecting triangles. Write them on your slates, that I may see whether you remember them all.

This having been done satisfactorily, let the pupils be called-upon to demonstrate any one of the problems: or rather, let each pupil, in turn, assign a problem, for solution, to the class. After this useful exercise, the master may proceed thus:

M.—What, do you think, should be our next step, after the investigation of *trilateral* figures?

P.—To investigate *quadrilateral* figures.

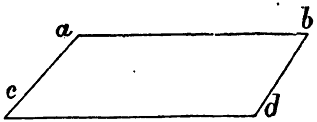
M.—State all you know of quadrilateral figures. (Lesson V. Introduction.)

Let the pupils repeat what they remember respecting them.

M.—Into what two groups may all quadrilateral figures be classed?

P.—Into parallelograms and trapeziums.

M.—We shall begin with parallelograms, and, first, consider the manner in which a parallelogram is constructed. Draw a parallelogram, and give a definition of it.



P.—If $a\,b$ is parallel to $c\,d$,
and $a\,c$ is parallel to $b\,d$,—
the figure $a\,b\,d\,c$ is a parallelogram.

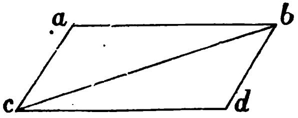
M.—Hence, a parallelogram is——

P.—A four-sided figure whose *opposite* sides are *parallel.*

M.—Well, one relation between the sides of parallelograms being known, you may be able to discover another: try.

P.—The opposite sides must be equal.

M.—Demonstrate this.



P.—Let $a\,b\,d\,c$ be a parallelogram,
its opposite sides shall be equal;
namely, $a\,b = c\,d$, and $a\,c = b\,d$.
Join $c\,b$:
$\because$ $a\,b$ is parallel to $c\,d$,
$\therefore$ $\angle\, a\,b\,c =$ alternate $\angle\, d\,c\,b$;
and $\because$ $a\,c$ is parallel to $b\,d$,
$\therefore$ $\angle\, a\,c\,b =$ alternate $\angle\, d\,b\,c$:
now, $c\,b$ is common to the triangles $a\,c\,b$ and $d\,b\,c$;
$\therefore$ $\triangle\, a\,c\,b = \triangle\, d\,b\,c$,
$a\,b = c\,d$, and $a\,c = b\,d$.

M.—Why are the triangles $a\,c\,b$ and $b\,d\,c$ equal to each other?

P.—Because they have two angles, and the side adjacent to them, of the one, *equal* to two angles, and the side adjacent to them, of the other, each to each.

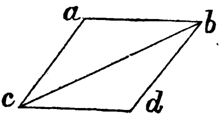
M.—And why, then, is $ab = cd$, and $ac = bd$?

P.—Because these are the sides which are *opposite* to the equal angles in the triangles.

M.—Then, conversely, if a four-sided figure has its opposite sides equal, what will you conclude as to the figure?

P.—That the figure is a parallelogram.

M.—Demonstrate this.



P.—Let $abdc$ be a four-sided figure whose opposite sides are equal: the figure is a parallelogram.

Join cb:

then, $\because ab = cd$, and $ac = bd$,

and cb is common to the triangles acb, bdc,

$\therefore \triangle acb = \triangle bdc$,

and $\angle abc = \angle dcd$.

But, these are alternate angles;

$\therefore ab$ is parallel to cd,

also $\angle acb = \angle dbc$.

Again, these are alternate angles;

$\therefore ac$ is parallel to bd:

and, hence, the figure $ab\,dc$ *is* a parallelogram.

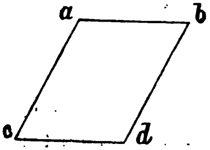
M.—Why are the triangles abc and dbc equal to each other?

P.—Because they have three sides of the one equal to three sides of the other, each to each.

M.—And, why is the angle abc equal to the angle dbc, and the angle acb equal to the angle dbc?

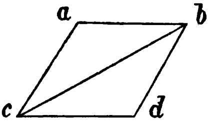
P.—Because these are opposite to the equal sides in the equal triangles.

M.—Again, if, in the four-sided figure $a\ b\ c\ d$, you knew that ab is equal and parallel to the opposite side cd, what would you conclude the figure to be?



P.—A parallelogram.

M.—Demonstrate this.



P.—Let $abdc$ be a four-sided figure of which the opposite sides ab, cd are equal and parallel: the figure shall be a parallelogram.

Join cd:
then $\because$ ab is parallel to cd,
$\therefore \angle abc =$ alternate $\angle dcb$,
and $ab = cd$:
and cb is common to the triangles abc, dcb—
$\therefore \triangle abc = \triangle dcb$,
and $\angle acb = \angle dbc$.
Now, these are alternate angles;
$\therefore$ ac is parallel to db,—
and $\therefore$ the figure $abdc$ is a parallelogram.

M.—Why is the triangle abc equal to the triangle dcb?

P.—Because these triangles have two sides of the one, *a b* and *b c*, equal to two sides of the other, *c d* and *b c*, each to each, and have likewise the angles *a b c* and *d c b*, contained by them, equal to each other.

M.—What else is known of the lines *a c* and *b d*, besides their parallelism?

P.—*a c* is equal to *b d*.

M.—Well, how have *a c* and *b d* been drawn?

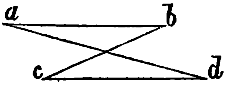
P.—Joining the extremities *a*, *c*, and *b*, *d*, of the parallel and equal straight lines *a b*, *c d*.

M.—Hence, the two straight lines which join the extremities of two equal and parallel straight lines—— Finish the sentence.

P.—Are equal and parallel.

M.—That is not *quite* correct. Repeat what has been said, and see if it be true in *every* case.

P.—No: for, in the annexed figure,



a d is *not* equal to *c b*,

and yet these two lines have been drawn so as to join the extremities of two equal and parallel straight lines.

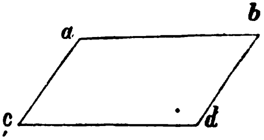
M.—Now, alter the preceding statement in conformity with this *finding.*

P.—The two straight lines which join the extremities of two equal and parallel straight lines *in the same direction*, or toward the same parts, are equal and parallel.

M.—What, then, is the *general* truth respecting the *sides* of parallelograms?

P.—The opposite sides of parallelograms are equal.

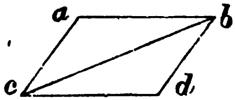
M.—Investigate the angles of a parallelogram.



P.—1. The sum of the interior angles is equal to four right angles.

2. The opposite angles are equal.

M.—Demonstrate these positions.



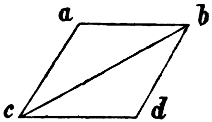
P.—Let $abdc$ be a parallelogram;
the sum of its *interior* angles shall be equal to *four right* angles.
Join bc:
then, $\because \angle$s of $\triangle\, abc = 2$ rt. $\angle$s,
and, also, $\angle$s of $\triangle\, dcb = 2$ rt. $\angle$s;
$\therefore \angle$s of the $\triangle$s $abc + dcb = 4$ rt. $\angle$s.
But, $\angle$s of $\triangle$s $abc + dcb$ make up the $\angle$s of the parallelogram $abdc$;
$\therefore$ the sum of the $\angle$s of a parallelogram $= 4$ rt. $\angle$s.
Their opposite angles are, also, equal to each other:
$\because$ it has been demonstrated that
$\triangle\, abc = \triangle\, dcb$,
and $\angle\, abc = \angle\, dcb$,
and $\angle\, acb = \angle\, dbc$;
$\therefore$ the whole $\angle\, abd =$ the whole $\angle\, dca$;
also, $\angle\, cab = \angle\, cdb$:
$\therefore$ the *opposite* angles of a parallelogram are equal to each other.

Let the pupils, now, be called-upon to demonstrate the last truth without the *auxiliary* line $c\,b$.

M.—The line $c\,b$ joining the vertices of the opposite angles is called a *diagonal.* What may be said of it?

P.—A diagonal divides a parallelogram into two *equal* parts.

M.—Demonstrate this.



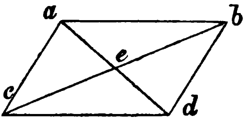
$\because a\,b = c\,d$, and $a\,c = b\,d$,
and $\angle\, c\,a\,b = \angle\, c\,d\,b$;
$\therefore \triangle\, c\,a\,b = \triangle\, c\,d\,b$,
and $\therefore$ the diagonal $c\,b$ divides the parallelogram into two equal parts.

M.—How many diagonals may be drawn in a parallelogram?

P.—Two.

M.—What may be said of them?

P.—The diagonals of a parallelogram bisect each other.



For, $\because a\,b = c\,d$,
and $\angle\, a\,b\,e = \angle\, d\,c\,e$,
and $\angle\, b\,a\,e = \angle\, c\,d\,e$;
$\therefore \triangle\, a\,b\,e = \triangle\, c\,d\,e$,
and $b\,e = c\,e$, and $a\,e = d\,e$.
Hence, the diagonals $a\,d$, $c\,b$ bisect each other in the point e.

M.—Mention some figures which are parallelograms.

P.—The square, the rhomb, and the rectangle.

M.—Since these are examples of the parallelogram, the truths you have demonstrated respecting parallelograms, *in general*, must likewise apply to a square, to a rhomb, and to a rectangle. But, these being *particular* cases of the parallelogram, each of them will give rise to *particular* truths. Begin with the square: define a Square.

P.—A square is a four-sided figure of which the sides are equal, and the angles right-angles.

M.—Now, what are the general truths common to all parallelograms?

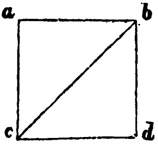
P.—1. Their opposite sides and angles are equal.

2. One diagonal bisects the parallelogram.

3. The two diagonals bisect each other.

M.—Now, find what *particular* truths apply to the Square.

P.—1. In a square, each diagonal bisects the opposite angles.



Thus, $\because ab = ac$,

$\therefore \angle abc = acb$:

But $\angle abc =$ alternate $\angle dcb$;

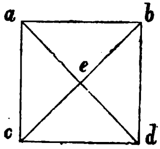
$\therefore \angle acb = \angle dcb$,—

and $\therefore \angle acd$ is bisected.

In the same way, it may be shown that the angle abd is bisected:

$\therefore$ the diagonal bc *bisects* the opposite angles.

Obs.—There are several other modes by which this truth may be demonstrated.



2. In a square two diagonals intersect each other at right angles; they are equal to each other; and the four segments are equal to each other.

For, $\because ab = ac$,
and ae is common to the triangles aeb, aec,
and $\angle cae = \angle bae$;
$\therefore \triangle aeb = \triangle aec$,
and $\therefore \angle aeb = \angle ace$,
which, accordingly, are rt. $\angle$s:
hence, all the angles at e are rt. $\angle$ s.
Also, $\because ab$ and $bd = ab$ and ac,
and $\angle abd = \angle bae$;
$\therefore$ the base $ad =$ the base bc:
and, since ad and bc bisect each other,
$\therefore ae = ed = eb = ec$.

M.—What axiom do you use here?

P.—Because the *whole* lines ad and bc are equal to each other, their *halves*, ae, ed, eb, and ec, are, likewise, equal to each other.

M.—Define a Rhomb.

P.—A rhomb is a four-sided figure whose sides are equal, but whose angles are not right angles.

M.—Now, ascertain the particular truths which apply to a rhomb.

P.—1. Each diagonal bisects the opposite angles.

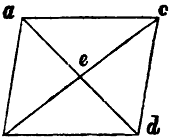
2. The two diagonals bisect each other at *right* angles.

3. The two diagonals are not equal to each other, nor are their four segments equal to each other.

Note.—Of these three positions, the former two are demonstrated as the similar cases respecting the square; at the proof of the third the pupils will easily arrive, guided by the *definition* of the rhomb.

M.—In what way, then, must two straight lines intersect each other, so that the figure, formed by the four lines joining their extremities, may be,

1. A square;
2. A rhomb?



P.—1. In order that the figure may be a *square*, the two straight lines must be equal, and must bisect each other at right angles.

Let ad be equal to cb,
and let them bisect each other at right angles in e;
the figure $acdb$ shall be a square.
$\because ae, be = ce, de,$ each to each,
and $\angle aeb = \angle ced$,
$\therefore$ base $ab =$ base cd,
also $\because be = ce$,
and ae is common to $\triangle$s aeb, aec,
and $\angle aeb = \angle aec$,
$\therefore ab = ac$,
$\therefore ab = ac = cd = db$:
and $\therefore$ the figure $acdb$ is *equilateral.*
Again, $\because \angle aec$ is a rt. $\angle$,
$\therefore \angle$s $eac + eca = 1$ rt. $\angle$.
But $\because ea = ec$;

$\therefore \angle eac = \angle eca$,
and $\therefore \angle eac$ is $\frac{1}{2}$ rt. $\angle$.
In a similar manner, it may be shown that $\angle eab$ is $\frac{1}{2}$ rt. $\angle$;
$\therefore \angle bac$ is a rt. $\angle$.
In the same way, it may be proved that $\angle$s acd, cdb, dba, are rt. $\angle$s:
$\therefore$ the figure $acdb$ is, likewise, *rectangular*;
and, having been proved to be *equilateral*,
$\therefore$ it is a *square*.

2. Again, in order that the figure may be a rhomb, the two straight lines must *not* be equal to each other, yet they must bisect each other at right angles.

(This the pupils will easily demonstrate.)

M.—If, then, one of the angles of a parallelogramic figure is a right angle, what are the other angles, necessarily?

P.—Likewise, right angles.

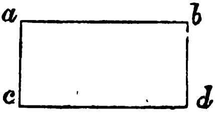
(This is *easily* demonstrated.)

M.—Again, if a four-sided figure is equilateral, is it, necessarily, equi-angular?

P.—No. (Instance—the square, and the rhomb.)

M.—But, if a four-sided figure be equi-angular, is it, necessarily, equilateral?

P.—No; it is, however, rectangular.



$\because$ the figure $abdc$ is equi-angular,
$\therefore$ each of its angles is *one-fourth* of their sum:
but the sum of the angles is 4 right $\angle$s;
$\therefore$ each of the angles is a *right* angle.

M.—And what, therefore, may be said of the *sides?*

P.—The opposite sides are parallel:
because ∠ *c a b* + ∠ *d b a* are two rt. ∠s,
∴ *a c* is parallel to *b d;*
and, for a similar reason, *a b* is parallel to *c d.*

M.—What, then, would you call a quadrilateral figure whose angles are equal to each other?

P.—A rectangular parallelogram [*a rectangle*].

M.—Is it always so?

P.—No; it may be a square: but, then, it must be known that the *sides* are *equal.*

M.—Now, find what particular truths apply to a rectangle.

P.—The two diagonals of a rectangle are equal to each other, and its four segments are equal to each other.

(Demonstration, analagous to that respecting the square and the rhomb.)

M.—In what way, then, must two straight lines intersect each other, that the figure, formed by joining their extremities, may be,

1. A rectangle;
2. A parallelogram?

P.—1. That the figure may be a rectangle, the two straight lines must be *equal* and bisect each other, yet not at right angles.

2. That it may be a parallelogram, the two straight lines must be *unequal* to each other, and must bisect each other, yet not at right angles.

(The demonstration is analagous to that respecting the square and rhomb.)

M.—We have, now, completed the investigation of parallelograms. What is the other group of quadrilateral figures which you mentioned?

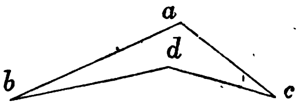
P.—Trapeziums [or, trapezia].

M.—Can any general truths be stated respecting trapeziums?

P.—No; unless some relation of the sides or angles be known.

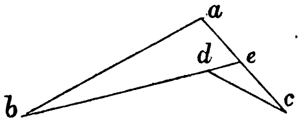
M.—There is a particular case of a trapezium—*A quadrilateral figure having three interior and one exterior angle.* Endeavour to describe such a figure; and, then, state what you can discover respecting it.

P.—



The figure $a\,b\,c\,d$ is such a trapezium; and we have proved before (Sect. 5.) that,

1. $ba + ac > bd + dc$;
2. $\angle\, bdc > bac$; and
3. $\angle\, bdc =$ interior $\angle$s $bac + abd + acd$.



Well, produce cd to e;

∴ ext. $\angle\, bdc$ of $\triangle\, dec =$ int. and opp. $\angle$s $bec + acd$.

But ext. $\angle\, bec$ of $\triangle\, bac =$ int. and opp. $\angle$s $bac + abd$;

∴ $\angle\, bdc =$ interior $\angle$s $bac + abd + acd$.

SUBSTANCE OF SECTION VI.

1. A parallelogram is a quadrilateral figure whose *opposite* sides are *equal.*

2. The opposite sides and angles of a parallelogram are equal to each other.

3. A quadrilateral figure whose opposite sides are equal is a parallelogram.

4. The two straight lines joining the extremities of two equal and parallel straight lines, toward the same parts, are themselves equal and parallel.

5. In a parallelogram, a diagonal bisects the parallelogram.

6. In a parallelogram, two diagonals bisect each other.

7. In a square, a diagonal bisects the opposite angles.

8. In a square, two diagonals are equal to each other; they intersect each other at right angles; and the four segments are equal to each other.

9. In a rhomb, each diagonal bisects the opposite angles.

10. In a rhomb, the two diagonals bisect each other at right angles.

11. If one of the angles of a parallelogram is a right angle, the other angles are, likewise, right angles.

12. If a quadrilateral figure is *equi*-angular, it is *rect*angular; that is, the figure is a rectangle.

13. The two diagonals, in a rectangle, are equal to each other.

14. In a trapezium having an exterior angle, the exterior angle is equal to the three interior angles.

SECTION VII.

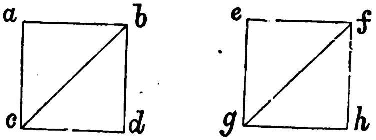
EQUALITY OF SQUARES, RECTANGLES, AND PARALLELOGRAMS.

M.—What must we know of a square in order to ascertain the number of square feet or square inches which it contains,—in short, its *area?*

P.—One of the sides; for, then, the other sides become known, as they are all equal to each other, and the angles are all right angles.

M.—When are two squares equal to each other?

P.—When they have one side of the one equal to one side of the other—or, when their bases are equal.



If $cd = gh$, the square $abdc$ shall be equal to the square $efhg$.

Join cd and gf:

$\because$ $abdc$ and $efhg$ are squares,

and $cd = gh$,

$\therefore bd = fh$;

and $\angle bdc = \angle fhg$,

$\therefore \triangle bdc = \triangle fhg$:

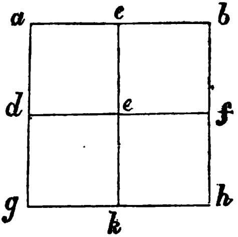
but $\triangle bdc$ is $\frac{1}{2}$ of the square $abdc$,

and $\triangle fhg$ is $\frac{1}{2}$ of the square $efhg$;
$\therefore$ square $abdc$ = square $efhg$.

M.—If, then, two equal straight lines be drawn, and upon them squares be constructed, what may be said of these squares?

P.—They are equal to each other.

M.—Draw a straight line on your slates, bisect it, and on the parts describe squares; likewise, describe a square on the whole line; then, *compare* these squares.



P.—The square upon the *whole* line ab is *four* times as great as the square upon *half* the line.

The square on half the line ab is one-fourth of the square on the whole line.

The two squares on half the line are, together, one-half of the square upon the whole line.

(The pupils should demonstrate what they have just-now stated.)

M.—Divide a straight line into *three* equal parts; describe squares on the whole line and on the parts, and *compare* them.

P.—The square on the whole line is *nine* times as great as the square on one-third of the line.

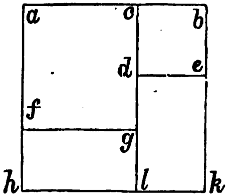
The square on one-third of the line is one-ninth part of the square on the whole line.

The three squares on the parts of the line are, together, one-third of the square of the whole line.

(Demonstration as in the preceding.)

The *comparison* of the square of the *whole* line with squares described upon *one-fourth*, *one-fifth*, &c. of the line, is now continued, as far as the master thinks expedient. The pupils are, then, required to compare these results with what they have learnt of square *numbers*. (Lessons on Number, p. 130. 2d Ed.)

M.—Again, if a straight line be divided into two *unequal*, instead of into two equal, parts, and squares be constructed upon these two parts, and likewise upon the whole line—what may be said of them?



P.—The square *a b k h* upon the whole line *a b* is greater than the square *a c g f* + *c b e d* upon the parts *a c* and *c b*, *by* the figure *d e k h f g*.

M.—You will be able to express the comparison better, if you produce *c g* to the side *k h*.

P.—The square upon the whole line is greater than the squares upon its parts, *by* the two figures *f g l k*, *d e k l*.

M.—And, of what kind are these figures?

P.—They are two rectangles.

M.—When may the area of a rectangle be known?

P.—When two of its adjacent sides are known; for, then, all its sides become known; and its angles are right angles.

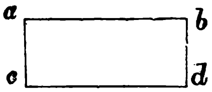
M.—Hence, when are two rectangles equal to each other?

P.—When two adjacent sides of the one are equal to two adjacent sides of the other, each to each.

M.—Now, find whether these two rectangles are equal to each other.

P.—They *are* equal to each other:
$\because ab = ah$, and $ac = af$,
$\therefore$ remainder $cb =$ remainder fh.
But $cb = de$;
$\therefore fh = de$:
also $\because bk = hk$,
and $be = de = kl$,
$\therefore$ remainder $hl =$ remainder ek,
$\therefore$ adjacent sides fh, $hl =$ adjacent sides de, ek;
and $\therefore$ the rectangles are equal to each other.

M.—Since two adjacent sides of a rectangle are sufficient toward ascertaining its area, it is usual, when speaking of a rectangle, as $abdc$, to say, "the rectangle contained by ab, ac," or, simply, "the rectangle ab, ac."



Now, observe by what lines the preceding two rectangles are contained.

P.—One of them by fg, fh, and the other by ek, ed.

M.—Compare these with ac, cb.

P.—fg, $fh = ac$, cb,
$\because fg = ac$, and $fh = cb$;
also ek, $ed = ac$, cb,
$\because ek = ac$, and $ed = cb$.

M.—Hence the square of ab is greater than the squares of ac and cb *by*—?

P.—*by* twice the rectangle contained by *a c, c b.*

M.—And, the square of *a b* is equal to —?

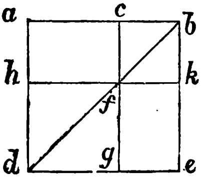
P.—the squares of *a c* and *c b,* together with twice the rectangle contained by *a c* and *c b.*

M.—Express this truth as a general proposition.

P.—If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

The master will, here, have a suitable opportunity of directing the attention of his pupils to a *numerical* truth analagous to the preceding—namely, that if a number be divided into any two parts, the square of the whole number is equal to the squares of the parts, together with twice the product of the parts; and, that, in general, the *product* of two numbers corresponds to a *rectangle* constructed by two straight lines.

M.—The same truth may be demonstrated by means of the following construction—
(*writing upon the slate*—)
Upon *a b* describe the square *a b e d;*
join *d b:*
from *c* draw *c f g* parallel to *b e* or *a d,*
and, through *f,* draw *k h* parallel to *a b* or *d e.*

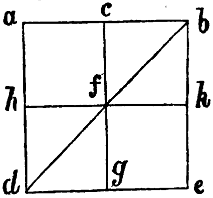


The pupils may either be left to demonstrate the proposition unaided, or the master may direct their attention to the following notes, which, for that purpose, he should write on the slate.

Show that,

1. The figure *c b k f* is a square;
2. The figure *h f g d* is a square, and equal to a square upon *a c;*
3. Rectangle *a c f h* = rectangle *f k e g:*
4. Thence, draw the necessary consequences.

The demonstration found by the pupils, with a slight exception as to the complements *af*, *fe*, is that in Euclid's El. B. II. P. 4; which see.



M.—Since, then, the rectangle *a c f h* is equal to the rectangle *f k e g*, what will be the result, if the square *c b k f* be added to each of them?

P.—The rectangle *a b k h* = the rectangle *c b e g.*

M.—And the sum of these two rectangles is, therefore, equal to what?

P.—To twice either of them.

M.—Now, what are the lines that *contain* the rectangle *a b k h?*

P.—The lines *a b* and *c b;* because *b k* = *c b.*

M.—Therefore, twice the rectangle contained by *a b* and *c b* is equal to what?

P.—To twice the square upon *c b*, together with twice the rectangle contained by *a c* and *c b.*

M.—And what will be the result, if, to each of the rectangles *a c f h* and *f k e g*, the square *h f g d*,—that is, the square upon *a c*,—be added?

P.—The rectangle *a c g d* = the rectangle *h k e d.*

M.—And, therefore, the sum of these rectangles is equal to —?

P.—To twice either of them.

M.—What are the lines that contain the rectangle $acgd$?

P.—The lines ab and ac; because $ad = ab$.

M.—Therefore, twice the rectangle contained by ab and ac is equal to what?

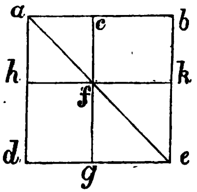
P.—To twice the square upon ac, together with twice the rectangle contained by ab and ac.

M.—Express this truth in words.

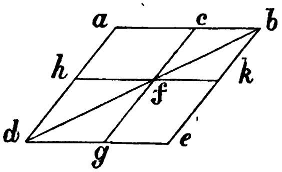
P.—If a straight line be divided into any two parts, twice the rectangle contained by the whole line and one of the parts is equal to twice the square upon that part, together with twice the rectangle contained by the parts.

M.—Two other interesting truths may be discovered from the preceding construction of the figure; and we shall, therefore, examine it more minutely.

A



B



For that purpose, I have constructed two figures, A and B—the former a square, the latter a parallelogram.

You have already shown, that the rectangles $acfh$, $fkeg$, in figure A, are equal to each other; now ascertain whether the same thing be true of the corresponding parallelograms in figure B.

P.—Yes: they are equal to each other.

$\because \triangle\, abd = \triangle\, bde$,

and $\triangle\, cfb = \triangle\, fkb$,—

and $\triangle\, hdf = \triangle\, dgf$;

$\therefore$ remaining parallelogram $acfh =$ remaining parallelogram $fkeg$.

M.—Now, tell me how the parallelograms $cbkf$ and $hfgd$ are situated with respect to the whole parallelogram $abed$.

P.—They are about the diagonal bd.

M.—And, to what are these, together with the two equal parallelograms $acfh$, $fkeg$, equal?

P.—They are together equal to the whole parallelogram $abed$.

M.—The two rectangles in figure A, or the two corresponding parallelograms in figure B, therefore, if added to the parallelograms which *are* about the diagonal of these figures, make-up or *complete* them, and are, on that account, called *Complements.* What may be said of them?

P.—The *complements* of the parallelograms which are about the diagonal of any parallelogram are equal to each other.

M.—And, what are the parallelograms which are about the diagonal of a square, in figure A?

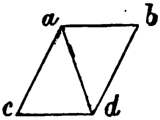
P.—The parallelograms which are about the diagonal of a square are, likewise, squares.

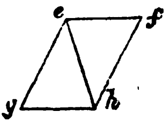
M.—When is a parallelogram known?

P.—When two of its adjacent sides and the angle contained by them are known; for, then, the other two sides are known and the remaining angles are known, because the opposite sides are parallel.

M.—Hence, when are two parallelograms equal to each other?

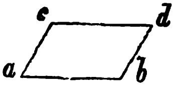
P.—When they have two adjacent sides, and the angle contained by them, of the one, equal to two adjacent sides, and the angle contained by them, of the other, each to each.

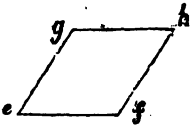




For, if ac, $cd = eg$, gh, each to each,
and $\angle\, acd = \angle\, egh$,
$\therefore \triangle\, acd = \triangle\, egh$.
But, $\triangle\, acd$ is $\frac{1}{2}$ of the parallelogram $abdc$,
and $\triangle\, egh$ is $\frac{1}{2}$ of the parallelogram $efhg$;
$\therefore$ these parallelograms are equal to each other.

M.—But if, of two parallelograms, it is known that the bases are equal to each other—are, then, the *parallelograms* equal?

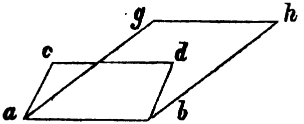




P.—No; for if $ab = ef$,
it is not necessary that the parallelogram $abdc$ should be equal to the parallelogram $efhg$.

M.—And, what happens, then, when the parallelogram $efhg$ is applied to the parallelogram $abdc$, so that the point e is upon the point a, and the base ef upon the base ab?

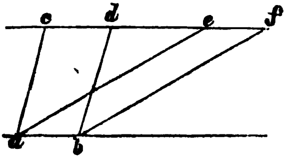
P.—The point *f* must fall upon the point *b*; because *e f* = *a b*.



M.—And, when will these parallelograms become equal?

P.—When *g h* is in the same straight line with *c d*; that is, when the parallelograms are between the same *parallels*.

M.—Describe two parallelograms in such a manner that they shall have a *common* base and be between the *same* parallels; and, then, determine whether they are equal.



P.—Let the parallelograms *a b d c*, *a b f e*, be upon the same base *a b*, and between the same parallels *a b*, *c f*; they shall be equal to each other.

∵ *c d* = *a b*, and *a b* = *e f*,

∴ *c d* = *e f*,

and ∴ *c e* = *d f*;

but, *a c* = *d b*,

and ∠ *b d f* = interior opposite ∠ *a c e*;

∴ △ *b d f* = △ *a c e*:

from the trapezium *a b f c* take the triangle *b d f* or *a c e*;

∴ parallelogram *a b d c* = parallelogram *a b f e*.

M.—Hence, parallelograms are equal when——

P.—They are upon the same base, and between the same parallels.

M.—What is to be observed with respect to the *distance* of parallel lines?

P.—Parallel lines are *equi*distant.

M.—And, in what way is their distance determined?

P.—By drawing a perpendicular.

M.—Hence, parallelograms which are between the same parallels have, likewise—what?

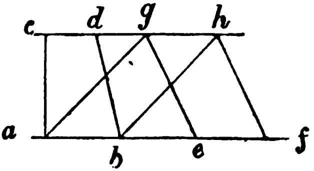
P.—The same perpendicular height; they have the same *altitude.*

M.—And, since parallelograms upon the same base and between the same parallels are equal to each other, what may be inferred of equal parallelograms which are upon the same base?

P.—Equal parallelograms upon the same base must be between the same parallels; or, they must have the same altitude.

M.—Instead of being upon the *same* base, the parallelograms may be——

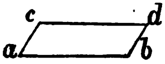
P.—Upon *equal* bases and between the same parallels.



Let the base $a\,b =$ the base $e\,f$,
and $c\,h$ be parallel to $a\,f$,

the parallelogram $abdc =$ parallelogram $efhg$.
Join ag, bh;
then $\because gh = ef$, and $ef = ab$,
$\therefore ab = gh$.
Also, ab is parallel to gh;
$\therefore ag$ is parallel and equal to bh,
and $\therefore abhg$ is a parallelogram;
and, it is equal to the parallelogram $abdc$,
$\because$ they are upon the same base ab, and
between the same parallels ab, ch:
and, for similar reasons,
the parallelogram $abhg =$ parallelogram $efhg$;
$\therefore$ parallelogram $abdc =$ parallelogram $efhg$.

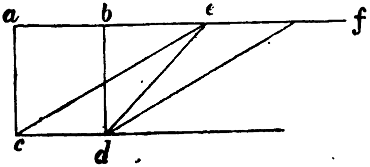
M.—It is usual to denote a parallelogram by two letters placed at the vertices of two opposite angles.

Thus, 

the parallelogram $abdc$ is usually denoted thus,—□ cb or □ ad.

What relation has a triangle to a parallelogram which is upon the same base and between the same parallels?

P.—If a triangle and a parallelogram are upon the same base and between the same parallels, the triangle is *one half* of the parallelogram, or the parallelogram is *double* the triangle.



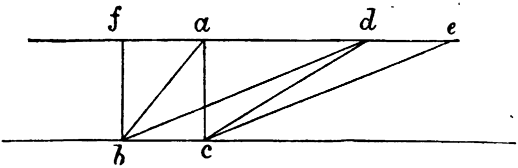
Let ▭ ad and △ cde be upon the same base cd,
and between the same parallels af, cd;
▭ ad shall be double of △ cde.
Draw df parallel to ce; hence, cf is a parallelogram:
and ∴ ▭ ad = ▭ cf,—
but ▭ cf is double of △ cde;
∴ ▭ ad is double of △ cde.

M.—Is it necessary that they should be upon the *same* base?

P.—No; the same thing is true of them if they be upon *equal* bases between the same parallels.

M.—From this and some of the preceding truths, I think, you will be able to discover a *fifth* instance of equality with respect to *triangles*.

P.—Yes; two *triangles* are equal when they are upon the same base and between the same parallels.



Let △s abc, bcd, be upon the same base, bc, and between the same parallels, fe, bc;
△ abc shall be equal to △ bcd.
Draw ce and bf parallel to bd and ac;
∴ ▭ fc = ▭ be;
but ▭ fc is double of △ abc,
and ▭ be is double of △ bcd:
∴ △ abc = △ bcd.

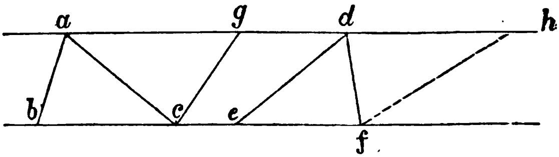
M.—In what does this fifth instance *differ* from the four previous instances of equality in triangles?

P.—In the previous instances, the sides and angles of the one triangle are equal to the sides and angles of the other triangle, each to each—whereas, in this case, the *areas* of the triangles are equal, yet not their respective sides or angles.

M. Well—in the previous instances, the triangles are said to be *identical.*

I think you will, now, be able to discover yet another instance of triangular equality.

P.—Yes; triangles are equal to each other when they are upon *equal* bases and between the same parallels.



Let $bc = ef$ and ah be parallel to bf;
$\triangle\, abc$ shall $= \triangle\, def$:
draw cg and fh parallel to ab and ed;
$\therefore \square\, bg = \square\, eh$;
but $\square\, bg$ is double of $\triangle\, abc$,
and $\square\, eh$ is double of $\triangle\, def$;
$\therefore \triangle\, abc = \triangle\, def$.

M.—Are these triangles identical?

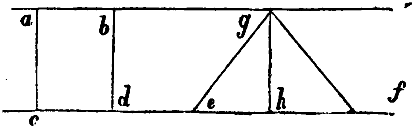
P.—They are identical only if

$$ab = de \text{ and } \angle\, abc = \angle\, def.$$

When this is not true, the areas of the triangles are equal, but the parts of the triangles are not identical.

M.—You have observed the circumstances under which a parallelogram is double of a triangle. Try, now, to find when a parallelogram is *equal* to a triangle.

P.—A parallelogram is equal to a triangle when both are between the same parallels, and the base of the triangle is double that of the parallelogram.

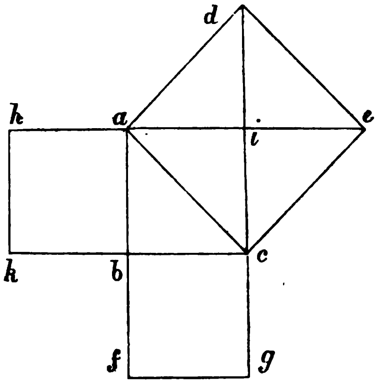


If *a g* is parallel to *c f*,
and *e f* is double *c d*,
▭ *a d* shall be equal to △ *e f g*;
bisect *e f* in *h* and join *g h*:
∴ △ *e h g* = △ *h f g*,
because they are upon equal bases and between the same parallels;
and ∴ △ *e f g* is double of △ *e h g*, or △ *h f g*:
but, ▭ *a d* is double of △ *e h g*, or △ *h f g*;
∴ ▭ *a d* = △ *e f g*.

M.—Again, when is a parallelogram double of another parallelogram?

P.—When they are between the same parallels, and the base of the one is double the base of the other. (Demonstration similar to that of the preceding position.)

M.—Draw an isosceles right-angled triangle, and upon its sides describe squares; then, compare these squares.



P.—1. The square upon $ab =$ square upon bc; because $ab = bc$.

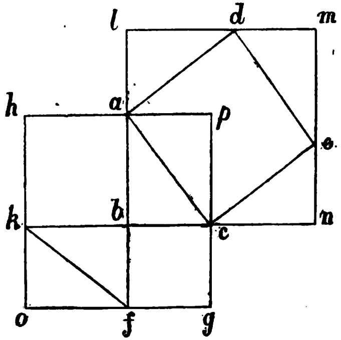
2. The square upon $ac =$ square upon $ab +$ the square upon bc;
or, the square upon ac is double the square upon ab or bc.
Join ae and dc;
then $\because$ $adec$ is a square,
$ai = ic$, and $\angle aic$ is a rt. $\angle$;
and $\therefore$ $\angle cai = \angle aci = \frac{1}{2}$ rt. $\angle$;
but $\angle bac = \angle bca = \frac{1}{2}$ rt. $\angle$;
$\therefore$ $\triangle abc = \triangle aic$,
because ac is common to them, and adjacent to equal angles;
and $\therefore$ $aicb$ is a square,
and the square $adec =$ the 4 $\triangle$s abc.

But, square $hb =$ square $aicb = 2\ \triangle$s abc,
and square $bg =$ square $aicb = 2\ \triangle\ abc$;
$\therefore$ squares $hb + bg = 4\ \triangle$s abc;
and $\therefore$ square $adec =$ squares $hb + bg$—
that is, the square upon $ac =$ *twice* the square upon ab or bc.

M.—Hence, what must be done in order to make a square equal to *twice* another square?

P.—Draw a right-angled isosceles triangle, whose equal sides are, each of them, equal to the side of the square. The square upon the side opposite the right angle is equal to the two squares upon the other sides of the triangle; and it is, therefore, equal to *twice* the other square.

M.—Draw any right-angled triangle, and construct squares upon its sides: then, compare these squares.



P.—The square upon $a\ c$ is equal to the squares upon $a\ b$ and $b\ c$ together.

Produce $b\ a$ and $b\ c$, and, through d and e, draw $l\ m$ and $m\ n$ parallel to $b\ n$ and $b\ l$, respectively;

also, produce $h a$ and $g c$ till they meet in p,
and produce $h k$ and $g f$ till they meet in o,
and join $k f$.

$$\left.\begin{array}{l} \therefore\ ln \text{ and } hg, \\ \quad pb \text{ and } bo, \end{array}\right\} \text{are parallelograms;}$$

and, $\because \angle\ ald$ is a rt. $\angle$,
$\therefore \angle$s $lad + lda =$ a rt. $\angle$,
and, $\therefore \angle$s $lad + lda = \angle\ dac$.
But, $\angle\ lda =$ alternate $\angle\ pad$;
$\therefore$ remaining $\angle\ lad = \angle\ pac$,
and rt. $\angle\ ald =$ rt. $\angle\ apc$,
and $ad = ac$;
$\therefore \triangle\ ald = \triangle\ apc$,
and $al = ap = bc$;
also $ld = pc = ab$.
And, in the same way, it may be shown,
that $\triangle\ abc = \triangle\ cne$,
and $ab = cn = ld$;
also, $bc = en = al$.
But, $\because lb = mn$, and $lm = bn$,
and $ld = cn$, and $al = en$;
$\therefore bc = md$, and $ab = me$,—
and $\therefore \triangle\ abc = \triangle\ dme$,—
and $\therefore lm = lb = bn = nm$;
and $\angle\ lbn$ is a rt. $\angle$;
$\therefore$ fig. ln is a square.
Again, $\because pc = ab = ha$,
and $eg = bc = ap$,
$\therefore pg = ph = ho = og$;
and $\angle\ pgo$ is a rt. $\angle$;

$\therefore$ fig hg is a square.

But, $hk = ab$, and $ap = al$;

$\therefore hp = bl$,

$\therefore$ square $hg =$ square ln:

also, $\because kb, bf = ab, bc$, each to each,

and rt. $\angle\, kbf =$ rt. $\angle\, abc$,

$\therefore \triangle\, kbf = \triangle\, abc$.

But, $\triangle\, kbf = kof$, and $\triangle\, abc = \triangle\, apc$;

$\therefore \triangle$s $abc + cne + emd + dla =$

$= \triangle$s $abc + apc + kbf + kof$.

From the square ln, take the triangles abc, cne, emd, and dla;

and from the square hg, take the triangles abc, apc, kbf, and kof:

$\therefore$ remaining square $dc =$ remaining square $hb +$ sq. bg.

The several parts of the preceding demonstration may be thus separated: show that

1. The triangles abc, ald, cne, and emd, are equal to each other.
2. That, therefore, ln is a square.
3. That hg is a square.
4. That hg is equal to ln.
5. That the triangles kbg, abc, kof, and apc, are equal to each other, and to the triangles abc, ald, dme, and cne.
6. And, therefore, that the square dc is equal to the square $hb +$ sq. bg.

M.—Express fully, in words, the truth you have demonstrated.

P.—In a right-angled triangle, the square described upon the side opposite to the right angle, is equal to the *sum* of the squares described upon the sides containing the right angle.

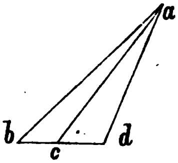
M.—If required to construct a square which shall be equal to two given squares, in what manner would you proceed?

P.—Draw two straight lines at right angles to each other, and equal to the bases of the given triangles respectively; the square upon the straight line joining their extremities shall, then, be equal to the two given squares.

M.—In an obtuse-angled triangle, squares being described upon the sides, it is required to compare the square described upon the side subtending the obtuse angle with the squares on the sides containing that angle.

The pupils should be left to find-out, unaided, the excess of this square above the sum of the other squares.

Should, however, the master have occasion to assist them, he may proceed thus:



M.—If △ *a c b* is obtuse-angled, at *c*, what must be done in order to make it a right-angled triangle?

P.—Produce *b c*, and, from *a*, draw *a d* at right angles to *b d*.

M.—Now, find whether the squares of *b d* and *a d* are greater or less than the squares of *b c* and *a c.*

P.—The square of *b d* is greater than the square of *b c*, *by* the square of *c d* and twice the rectangle contained by *b c*, *c d ;*
and the square of *a d* is less than the square of *a c*, *by* the square of *c d :*
therefore, the squares of *b d* and *a d* are, together, greater than the squares of *b c* and *a c*, *by* twice the rectangle contained by *b c*, *c d.*

M.—And, to what square are the squares of *b d* and *a d*, together, equal?

P.—To the square of *a b.*

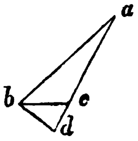
M.—Draw, thence, the necessary inference.

P.—The square of *a b* is, therefore, greater than the squares of *b c* and *a c* *by* twice the rectangle contained by *b c*, *c d.*

Exhibit this demonstration on your slates; and, instead of writing "the square of *a b*," write $a b^2$.

P. $b d^2 > b c^2$, *by* $c d^2$ and twice the rectangle $b c$, $c d$,
and $a d^2 < a c^2$, *by* $c d^2$ (because $a c^2 = a c^2 + c d^2$);
$\therefore b d^2 + a d^2 > b c^2 + a c^2$, *by* twice the rectangle $b c$, $c d$:
but $b d^2 + a d^2 = a b^2$;
$\therefore a b^2 > b c^2 + a c^2$, *by* twice the rectangle $b c$, $c d$.

M.—Instead of producing *b c*, what else might have been done in order to make the obtuse-angled triangle, *a b c*, a right-angled triangle?



P.—The side *a c* might have been produced, and from *b* a perpendicular been drawn to it.

M.—Try whether, in this case, the demonstration would be the same.

[The pupils will find it would.]

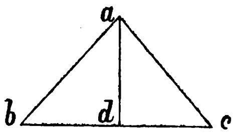
M.—Express, in words, the truth you have demonstrated.

P.—In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, *by* twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and, the straight line intercepted, without the triangle, between the perpendicular and the obtuse angle.

Obs.—While it is, of course, not to be expected, that the pupils can express the result of a demonstration in terms so accurate and precise as the preceding—yet, the teacher is earnestly counselled to require them, in every case, to state, in their *own* words, the truths which they have discovered, — altering and amending their expressions till they assume the form of a strict and regular proposition: for, they will, thus, be led to perceive clearly the necessary conditions and exact limits of the results of their investigations, and to acquire a readiness in embodying them in correct and suitable phraseology. Perhaps, indeed, of all the various exercises arising out of geometrical study, few tend more than this toward *the general improvement of the mental faculties,*—which, be it recollected, is a principal object of this Treatise.

M.—Draw any acute-angled triangle, and compare the square described upon any one of its sides with the sum of the squares described upon the other two sides.

If the master should find it necessary to assist his pupils, he may proceed thus:



M.—Let abd be a triangle right-angled at d; what must be done in order to make it an acute-angled triangle?

P.—Produce bd to any point, c,—and draw ac.

M.—Now, compare bc^2 with bd^2.

P. bc^2 is greater than bd^2 *by* dc^2 and twice the rectangle bd, dc.

M.—Also, compare ac^2 with ad^2.

P. ac^2 is greater than ad^2 *by* dc^2.

M.—Hence, $bc^2 + ac^2$ are greater than $bd^2 + ad^2$, by what—?

P.—By twice dc^2 and twice the rectangle bd, dc.

M.—But, to what square are $bd^2 + ad^2$ equal?

P.—To ab^2.

M.—Therefore, $bc^2 + ac^2$ are greater——Finish the sentence.

P.—Than ab^2, *by* twice dc^2 and twice the rectangle bd, dc.

M.—And, what figures are formed by the two squares of dc and twice the rectangle bd, dc? (p 237, No. 3.)

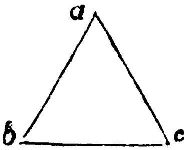
P.—Two rectangles, which are equal to twice the rectangle contained by bc and cd.

M.—And, therefore, $bc^2 + ac^2$ are greater than ab^2 *by* —?

P.—By twice the rectangle bc, cd.

M.—Or ab^2 is —?

P. ab^2 is less than $bc^2 + ac^2$ *by* twice the rectangle bc, cd.

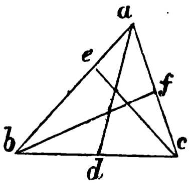


M.—If, then, in the acute-angled triangle, abc, it should be required to compare ab^2 with $bc^2 + ac^2$, how would you proceed?

P.—From a, we would draw a straight line at right angles to bc; or, from b, would draw a line at right angles to ac.

M.—And, if ac^2 is to be compared with $ab^2 + ac^2$; or bc^2 with $ab^2 + bc^2$ —?

Note.—It is advisable that the pupils should actually draw the perpendiculars, and write upon their slates the relations which the squares bear to one another, and fully demonstrate each case. Thus, they will find that



1. $ab^2 < bc^2 + ac^2$ by $2\ bc,\ cd$, or by $2\ ac,\ cf$.
2. $ac^2 < ab^2 + bc^2$ by $2\ bc,\ bd$, or by $2\ ab,\ be$.
3. $bc^2 < ab^2 + ac^2$ by $2\ ac,\ af$, or by $2\ ab,\ ae$.

Obs.—If $\angle\ acb$ be an obtuse angle,
the perpendicular ad will fall *without* $\triangle\ abc$;
and, then, $ac^2 < ab^2 + bc^2$ *by* $2\ bc,\ bd$,—as previously.

M.—Now, express, in words, the truth you have demonstrated.

P.—In acute-angled triangles, the square of the side subtending any of the acute angles is less than the squares of the sides containing the same angle, by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular, drawn to it from the opposite angle, and the acute angle.

SUBSTANCE OF SECTION VII.

1. Squares are equal to each other when their *bases* are equal.

2. If a straight line be divided into any two parts, the square of the whole line is equal to the squares

of the two parts, together with twice the rectangle contained by the parts.

3. If a straight line be divided into any two parts, twice the rectangle contained by the whole line and one of the parts is equal to double the square of that part, together with twice the rectangle contained by the parts.

4. The complements of the parallelograms, which are about the diagonal of any parallelogram, are equal to each other.

5. The parallelograms which are about the diagonal of a square are, likewise, squares.

6. Parallelograms and triangles which are upon the *same* base and between the same parallels are equal to each other.

7. Parallelograms and triangles which are upon *equal* bases and between the same parallels are equal to each other.

8. If a parallelogram and a triangle are upon the same base, or upon equal bases, and between the same parallels, the parallelogram is double the triangle.

9. In a right-angled triangle, the square of the side subtending the right angle is *equal* to the sum of the squares of the sides containing the right angle.

10. In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is *greater* than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced,

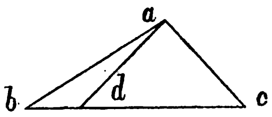
the perpendicular falls and the straight line intercepted, without the triangle, between the perpendicular and the obtuse angle.

11. In acute-angled triangles, the square of the side subtending any of the acute angles is *less* than the squares of the sides containing the same angle, by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular, drawn to it from the opposite angle, and the acute angle.

SECTION VIII.

PROPORTIONAL TRIANGLES.

M.—Describe any triangle, and, from one of its angles, draw a straight line to the opposite side: what figures are thus obtained?



P.—Two triangles, *a b d*, and *a d c.*

M.—When are these two triangles equal to each other?

P.—When the straight line, *a d*, bisects the base, *b c;* because, then, the triangles have equal bases, *b d* and *d c*, and they are between the same parallels.

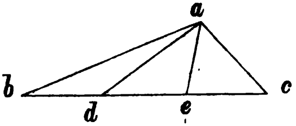
M.—And, what part of the whole triangle *a b c* is each of the triangles *a b d*, *a d c?*

P.—*One-half* of the triangle *a b c.*

M.—And, what part is the base *b d* of the whole base *b c?*

P.—Also, one-half.

M.—How must *a d* be drawn, so that the triangle *a b d* may be one-third of the whole triangle *a b c?*



P.—The base *b c* must be divided into three equal parts, *b d*, *d e*, and *e c*, and, then, *a d* must be drawn.

M.—Demonstrate that the triangle *a b d* is *one-third* of the triangle *a b c*.

P.—Join *a e;*

then $\because bd = de = ec$,

$\triangle abd = \triangle ade = \triangle aec$;

and $\therefore \triangle abd$ is one third of $\triangle abc$.

M.—Hence, when two triangles are between the same parallels, but the base of the one is one-half of the base of the other, what may be said of these triangles?

P.—The one is one-half of the other.

M.—And, when the base of the one is one-third of the base of the other —?

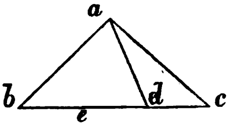
P.—The one is then one-third of the other triangle.

M.—Under what circumstances must *a d* be drawn, so that the triangle *a b d* may be one-fourth, one-fifth, one-sixth, &c. of the whole triangle *a b c?*

P.—The base *b c* must be divided into four, five, six, &c. equal parts, of which *b d* shall be one.

M.—And, how must *a d* be drawn so that the triangle *a b d* may be two-thirds of the whole triangle *a b c?*

P.—The base *b d* must be made two-thirds of *b c;* and this is done by dividing *b c* into three equal parts,



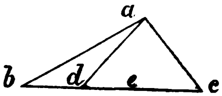
b e, *d e*, and *d c*. Draw *a d;*

△ *a b d* is two-thirds of △ *a b c*.

M.—How must *a d* be drawn so that the triangle *a b d* may be three-fourths, four-fifths, five-sixths, two-sevenths, &c. of the whole triangle *a b c?*

P.—The base *b d* must be made the *same part,* three-fourths, four-fifths, &c. of the base *b c*.

M.—How must the base *b c* be divided, in order that one of the resulting triangles may be one-half of the other?



P.—Divide *b c* into three equal parts, *b d*, *d e*, *e c*, and join *a d;*

△ *a b d* is ½ of △ *a d c*.

M.—We may, then, say that the triangle *a b d* is to the triangle *a d c* as—what two *lines?*

P.—As *b d* to *d c*.

M.—Or, as what two *numbers?*

P.—As 1 to 2.

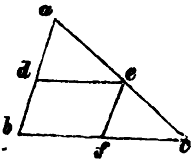
M.—Now, draw *ad* so, that the triangles may be to each other as 1 to 3; then, as 1 to 4, 1 to 5, &c.—2 to 3, 3 to 4, 4 to 5, &c.

The pupils having done so, and having *demonstrated* each case, the master may ask,

M.—Hence, how are triangles related to each other, when they are between the same parallels?

P.—Triangles between the same parallels are to one another as their *bases.*

M.—Draw any triangle, bisect one of its sides, and from the point of bisection draw a straight line parallel to either of the other two sides. Then, find whether that side be likewise bisected.

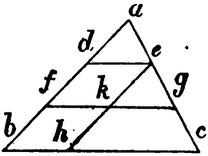


P.—Let $ad = db$, and de be parallel to bc;
then shall $ae = ec$.
Draw ef parallel to ab,
$\therefore$ df is a parallelogram,
and $ef = bd = ad$;
also, $\angle efc =$ int. opp. $\angle dbf = \angle ade$,
and, $\angle ecf = \angle dea$;
$\therefore$ $\triangle ade = \triangle efc$,
and, $\therefore ae = ec$,
that is, ac *is likewise bisected.*

M.—Well, what part of the whole triangle abc is the triangle ade, thus cut-off by the parallel line de?

P.—One-fourth of the triangle abc:
for, the parallelogram df is double the $\triangle ade$,
and $\triangle efc = \triangle ade$;
$\therefore$ whole $\triangle abc = 4 \times \triangle ade$,
that is, $\triangle ade$ is $\frac{1}{4}$ of $\triangle abc$.

M.—Now, draw de, from the third part of ba, parallel to bc; and find what part ae, then, is of the whole ac.



P.—Let $ad = df = fb$;
draw de and fg parallel to bc,
and ekh parallel to ab:
$\therefore ae = eg$,—because $ad = df$, and de is parallel to fg;
also, $ek = kh$,—because $df = fb$, and dh is a parallelogram;
and $\therefore eg = gc$.
Hence, $ae = eg = gc$;
that is, ae is, likewise, one-third of ac.

M.—And, what part of the whole triangle is the triangle ade, thus cut-off?

P.—One-ninth part.
$\because \triangle afg = \triangle ehc$,
and $\triangle afg = 4 \times \triangle ade$,
$\therefore \triangle efc = 4 \times \triangle ade$:
also, parallelogram $df = 4 \times \triangle ade$;
$\therefore$ trapezium $dbce = 8 \times \triangle ade$,
and $\therefore \triangle abc = 9 \times \triangle ade$;
that is, $\triangle ade$ is *one-ninth* part of $\triangle abc$.

The master should, now, desire his pupils to draw a parallel from *one-fourth*, *one-fifth*, *one-sixth*, *&c. part* of *a b*, and then to find whether, 1, the side *a c* is divided by this parallel in a similar manner; and 2, what portion of the whole triangle is the triangle thus cut-off.

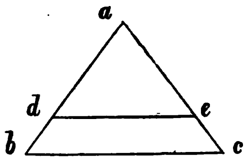
The pupils should find that,

1. The side *a c* is, in every case, divided similarly to the side *a b*.

2. That, if *a d* is one-half, or one-third, or one-fourth, or one-fifth, &c. of *a b*,

△ *a d e* is *one-fourth*, or *one-ninth*, or *one-sixteenth*, or *one-twenty-fifth*, &c. of △ *a b* c.

M.—Hence, we shall be able to ascertain several very important truths.



For this purpose we shall draw, from any point *d*, in the side *a b* of the triangle *a b c*, a straight line *d e* parallel to the side *b c*,—and then inquire,

1. In what manner the side *a c* is divided.

P.—As *a b* is divided;

that is, what part soever *a d* is of *a b*,

the same part is *a e* of *a c*;

and, what part soever *b d* is of *a b*,

the same part is *e c* of *a c*.

M.—When such a relation exists between four *numbers*, what may be said of them? (Lessons on Number, p. 218, 2nd. Ed.)

P.—They are *proportional* to each other.

M.—Now, since a similar relation exists between the four lines ad, ab, and ae, ac, what may, likewise, be said of them?

P.—These lines are, likewise, proportional to each other.

M.—You know the usual way of designating quantities as proportional to each other; now, express the relation which these lines bear to each other, in the same way.

P. $ad : ab :: ae : ac$.

M.—And, since a similar relation exists between the lines db, ab, and ec, ac, you would express this —?

P. $db : ab :: ec : ac$.

M.—Now, compare these two proportions with each other, and ascertain what inference may be thence drawn.

P.—Since $ad : ab :: ae : ac$,

and $db : ab :: ec : ac$,

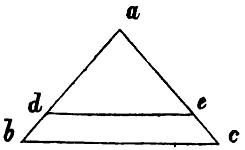
$\therefore\ ad : db :: ae : ec$.

M.—What axiom have you used in drawing this inference?

P.—That, two lines which have the same proportion to two other lines that other two lines have, are proportional to each other.

M.—And, what may be said generally, if from any point in the side of a triangle a straight line be drawn parallel to either of the other sides?

P.—It will divide these sides *proportionally.*

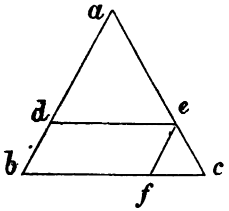


M.—*Secondly,* we shall compare the triangles *a b c* and *a d e.* What have you to observe of them?

P.—The angles of △ *a b c* are equal to the angles of the triangle *a d e,* each to each: that is,
∠ *b a c* is common to each of them;
and ∠ *a b c* = ∠ *a d e,* and ∠ *a c b* = ∠ *a e d.*

M.—Triangles which are such, that the angles of the one are equal to the angles of the other, each to each, are said to be *similar* triangles. Hence, a line drawn parallel to one of the sides of a triangle cuts-off—— ?

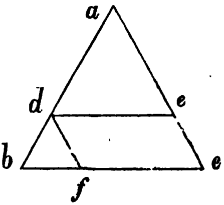
P.—A triangle which is *similar* to the whole triangle.



M.—If, then, *e f* be drawn parallel to *a b,* what must the triangle *e c f* be, when compared with the whole triangle *a b c?*

P.—Likewise, similar to it; and, therefore, also, similar to the triangle *a d e.*

M.—*Thirdly*, compare the sides of the similar triangles abc and ade.



P. $\because de$ is parallel to bc,
$\therefore ad : ab :: ae : ac;$
and $\because df$ is parallel to ac,
$\therefore ad : ab :: cf : bc.$
But $cf = de;$
$\therefore ad : ab :: de : bc.$
Again, $ad : ab :: ae : ac;$
$\therefore ae : ac :: de : bc.$

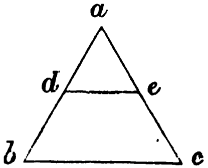
M.—Express this relation of the sides of similar triangles, in words.

P.—In similar triangles, the sides which are opposite to equal angles are proportional to each other.

M.—In similar triangles, the sides which are opposite to equal angles are called the *homologous* sides. What are the homologous sides in these triangles ?

[The pupils name them.]

M.—*Fourthly*, compare the areas of the triangles ade and abc.



P.—If we knew what part *a d* is of *a b*, we could tell what part the triangle *a d e* is of the whole triangle *a b c*.

M.—Suppose *a d* to be *three-fifths* of *a b*.

P.—Then the triangle *a d e* is *nine-twenty-fifths* of the triangle *a b c*.

M.—And, how are nine-twenty-fifths related to three-fifths?

P.—Nine-twentyfifths are the square of three-three-fifths.

M.—Hence, if *a d* is any part whatever of *a b*, *what part* will the triangle *a d e* be of the triangle *a b c?*

P.—The part expressed by the *square* of *a d*.

M.—And, by what name did we denote the sides of similar triangles which are opposite to equal angles?

P.—They are called the *homologous* sides.

M.—Now, express, in words, the relation which similar triangles have to one another.

P.—Similar triangles are to one another as the *squares* of their homologous sides.

SUBSTANCE OF SECTION VIII.

1. Triangles between the same parallels are to one another as their *bases*.

2. If, from any point in any side of a triangle, a straight line be drawn parallel to either of the other sides, the segments of the sides are proportional to each other.

3. Triangles are said to be similar to each other, when the angles of the one are equal to the angles of the other, each to each.

4. In similar triangles, the sides which are opposite to the equal angles are called *homologous* sides.

5. In similar triangles, the homologous sides are proportional to each other.

6. The areas of similar triangles are to one another as the squares of their homologous sides.

SECTION IX.

POLYGONS.

The aim of this paragraph is, to afford a suitable occasion and method for recapitulating and applying the foregoing truths, and, likewise, to excite the pupils to discover some of the more obvious and important truths arising from the investigation of regular polygons. In order to accomplish this object, the master should abstain from giving them any assistance, and be satisfied if their endeavours are successful in part only,—rather aiming to obtain little, [*but, that little from the pupils themselves,*] than dragging them, as it were, forcibly forward, to wrestle with new difficulties.

The following are some of the results *actually* obtained, in this manner, from the author's pupils.—

M.—We have, now, arrived at the investigation of polygons, and it is my wish that you should endeavour to discover some, at least, of their properties. In what way would you begin the investigation of a regular *pentagon*, for instance? What questions would you propose to yourselves, concerning it?

P.—1. We would endeavour to find the sum of its angles.

2. Then, each angle separately.

3. Each of the sides may, next, be produced, and the sum of the exterior angles found.

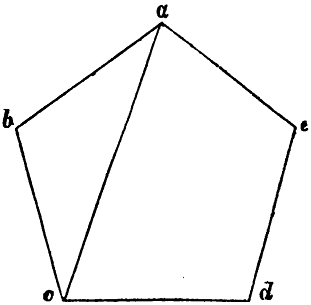
4. The sides may, then, be produced till they meet, and the sum of the angles be ascertained, which these sides form at the point of concourse.

5. One diagonal may be drawn, and the relation it bears to the other sides be investigated.

6. Two, three, four diagonals may be drawn, and their relations to each other be investigated.

These answers, which the master *may* obtain from his pupils, he should write upon the large school-slate, and, directing their attention to their own observations, leave them to demonstrate each satisfactorily.

1. In a regular pentagon, the sum of the angles = 6 rt. ∠s.



Join $a\,c$:

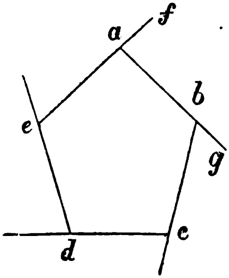
$\because$ ∠s of $\triangle\, a\,b\,c = 2$ rt. ∠s,

and ∠s of trapezium $a\,e\,d\,c = 4$ rt. ∠s,

$\therefore$ ∠s of the pentagon $= 2 + 4 = 6$ rt. ∠s.

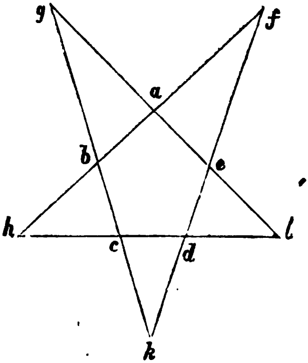
2. Each of the angles is, therefore, one-fifth of 6 rt. ∠s.

i. e. six-fifths of a rt. ∠, or one rt. ∠ and one-fifth of a rt. ∠,—and, therefore, obtuse.



3. The sum of the exterior angles = 4 rt. ∠s. For, ∵ each ext. ∠ *f a b* + its adjacent int. ∠ *b a e* = 2 rt. ∠s,

∴ all the ext. ∠s + all the int. ∠s = 10 rt. ∠s.
But, the int. ∠s of the pentagon = 6 rt. ∠s;
∴ ext. ∠s = 4 rt. ∠s.



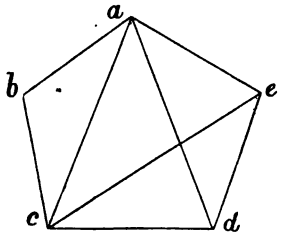
4. If the sides of a regular pentagon be produced till they meet, the sum of the angles at the points of concourse = 2 rt. ∠s.

For, $\because$ ext. $\angle\, lek$ of $\triangle\, egk =$ int. opp. $\angle$s $egk + ekg$,
and, likewise, ext. $\angle\, lde$ of $\triangle\, dfk =$ int. opp. $\angle\, dfh + dhf$,

$\therefore$ $\angle$s $lek + lde = 4$ $\angle$s at g, k, f, h.

To each of these equals add $\angle$ at l;

$\therefore$ $\angle$s of $\triangle\, edl = 5$ $\angle$s at g, k, f, h, l; *i.e.* $= 2$ rt. $\angle$s.



5. A diagonal, *ac*, is parallel to the side, *ed*.

Join *ad*, *ce*:

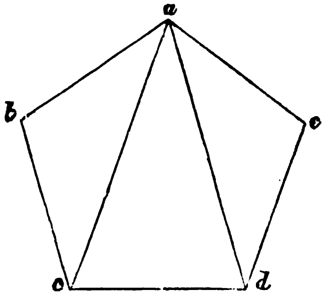
then $\because$ $ae, ed = cd, ed$, each to each,

and $\angle\, aed = \angle\, cde$,

$\therefore$ $\triangle\, aed = \triangle\, cde$.

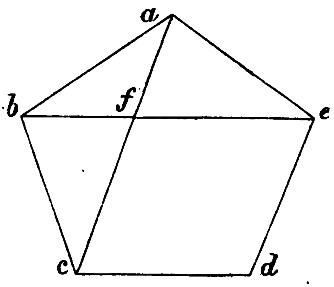
But, equal $\triangle$s upon the same base are between the same parallels;

$\therefore$ ac is parallel to ed.



6. If, from the same angular extremity, two diagonals *a c*, *a d*, be drawn, they shall be equal to each other; and each of the angles *a c d*, *a d c*, is double the angle *c a d*.

For, $\because$ *b a*, *b c* $=$ *e a*, *e d*, each to each,
and $\angle$ *a b c* $=$ $\angle$ *a e d*,
$\therefore$ base *a c* $=$ base *a d*,
and $\therefore$ $\angle$ *a c d* $=$ $\angle$ *a d c*;
also, $\because$ whole $\angle$ at *a* $=$ whole $\angle$ at *d*,
and $\angle$ *e a d* $=$ $\angle$ *e d a*,
$\therefore$ the remaining $\angle$ *b a d* $=$ the remaining $\angle$ *a d c*.
But, $\because$ *a d* $\parallel$ *b c*,
$\therefore$ $\angle$ *b c a* $=$ alt. $\angle$ *c a d* $=$ $\angle$ *b a c*;
and $\therefore$ $\angle$ *b a d* is double of $\angle$ *c a d*.
But $\angle$ *a d c* $=$ $\angle$ *b a d*;
$\therefore$ $\angle$ *a d c* is, likewise, double of $\angle$ *c a d*.



7. If, from two angular extremities *a*, *b*, diagonals *ac*, *be*, be drawn,—

1. The figure *f c d e* is a parallelogram;
2. The greater segment *f c* is equal to a side of the regular pentagon; and the figure *f c d e* is, therefore, a *rhomb*,

3. $f b a$ is similar to $\triangle\, a b e$.

1. $\because a c \parallel e d$ and $b e \parallel c d$,

$\therefore$ fig. $f c d e$ is a parallelogram ; and,

2. $\therefore f c = e d = f e = c d$,

a side of the pentagon ;

the fig. $f c d e$ is, therefore, a *rhomb :*

3. also $\because$ the opposite $\angle$s of parallelograms are equal to each other,

$\therefore \angle c f e = \angle c d e = \angle b a e$;

but, $\angle a f b = \angle c f e$;

and $\therefore$, also, $\angle a f b = \angle b a e$,

and $\angle a b f$ is common to $\triangle$s $a b e$, $a b c$;

$\therefore$ remaining $\angle b a f =$ remaining $\angle a e b$,

and $\therefore \triangle\, a b f$ is similar to $\triangle\, a e b$.

From this it follows that,

1. $f a = f b$,

$\because \angle a e b = \angle a b e = \angle b a f$;

and $\therefore$ rectangle $c f \times f a =$ rectangle $b f \times f e$:

2. also $\because \triangle$s $a b e$, $a b f$, are similar,

$\therefore b e : b a :: b a : b f$,

and $\therefore$ rectangle $b e \times b f = b a^2$.

But, $\because f e = b a = f e$,

$\therefore b a^2 = f e^2$,

and $\therefore$ rectangle $b e \times b f = f e^2$.

Obs.—The master has, here, an opportunity of explaining what is meant by "dividing a straight line into *extreme and mean ratio.*"

The pupils may, now, be required to investigate, in a similar manner, a regular hexagon.

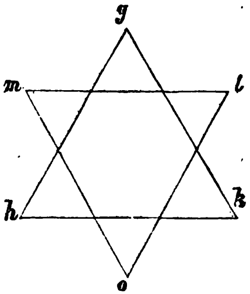
They will find that,

1. The sum of the interior angles of a regular hexagon is equal to *eight* right angles.

2. Each of the angles is, therefore, one-sixth of eight right angles, [*i. e.* four-thirds of a rt. $\angle$ = one and one-third of a rt. $\angle$,] and is, therefore, obtuse.

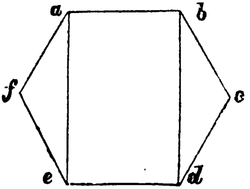
3. If each of the sides be produced, the sum of all the exterior angles is equal to four right angles.

4. If the sides be produced till they meet, the sum of the angles at the points of concourse = 4 rt. $\angle$s.



For, $\because$ $\angle$s of $\triangle\, g\,h\,k = 2$ rt. $\angle$s,
and, likewise, $\angle$s of $\triangle\, l\,m\,o = 2$ rt. $\angle$s,
$\therefore$ $\angle$s at the points $g, h, k, l, m, o = 4$ rt. $\angle$s.

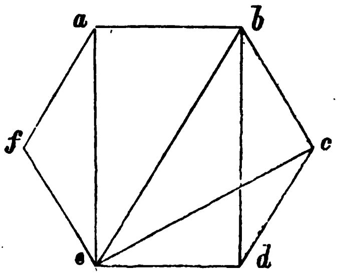
5. The opposite sides are parallel to each other.



Join $a\,e$, $b\,d$
then $\because$ $f\,a, f\,e = c\,b, c\,d$, each to each,

and ∠ at *f* = ∠ at *c*,
∴ *a e* = *b d*.
But *a b* = *e d*;
∴ figure *a b d e* is a parallelogram,
and ∴ *a b* ∥ *e d*.

6. Also ∵ whole ∠ at *a* = whole ∠ at *e*,
and ∵ ∠ *f a e* = ∠ *f e a*,
∴ remaining ∠ *b a e* = remaining ∠ *d e a*.
But ∵ *a e* ∥ *e d*;
∴ ∠s *b a e* + *d e a* = 2 rt. ∠s;
∴ ∠ *b a e* is a rt. ∠,
and ∴ fig. *a d* is a rectangle.

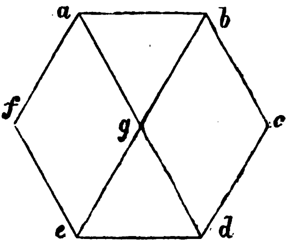


7. A diagonal, *b e*, is
1. parallel to *c d* or *a f*;
2. bisects the angles at *b* and *e*, and, also, the hexagon.

1. Join *c e*, *b d*;
and ∵ *b c*, *c d* = *e d*, *d c*, each to each,
and ∠ at *c* = ∠ at *e*,
∴ △ *b c d* = △ *e d c*:

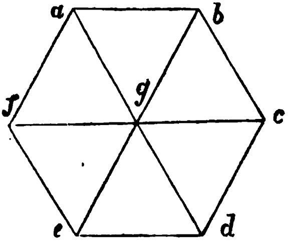
and they are upon the same base dc;
$\therefore$ $be \parallel cd$, and $\therefore$, likewise, $\parallel af$.

2. Also, join ae:
$\because$ in $\triangle$s bae, bce,
rt. $\angle\, bae =$ rt. $\angle\, bce$,
and $ab = bc$,
$\therefore$ $\triangle\, bae = \triangle\, bce$,
and $\therefore$ $\angle\, abe = \angle\, cbe$.
But $\angle\, abe =$ alternate $\angle\, bed$;
$\therefore$ the $\angle$s at b and e are bisected.
Again, $\because$ $\triangle\, abe = \triangle\, bce$,
and $\triangle\, afe = \triangle\, edc$,
$\therefore$ $\triangle$s $abe + afe = \triangle$s $bce + edc$,
and $\therefore$ be bisects the hexagon.



8. If two diagonals be, ad, be drawn, each of the $\angle$s age, $bgd =$ the angle of the hexagon.
For, $\because$ $ad \parallel fe$ and $be \parallel af$,
$\therefore$ fig. ae is a parallelogram,
and $\therefore$ $\angle\, age =$ opp. $\angle\, afe$;
also $\because$ $ag = fe$ and $af = ge$:
$\therefore$ fig. ae is a rhomb;

and each of the segments of the diagonals is equal to a side of the hexagon.



9. A third diagonal passes through the point of intersection of the other two.

Join *f g*, *g c*;
and ∵ fig. *a e* is a rhomb,
∴ ∠ *a g f* = ∠ *e g f*.
Similarly, ∠ *b g c* = ∠ *d g c*,
and ∠ *a g b* = vert. ∠ *e g d*;
∴ ∠s *a g f* + *a g b* + *b g c* = ½ the sum of the ∠s about the point *g*, = 2 rt. ∠s,
∴ *f c* is a straight line;
thus, three diagonals intersect each other in the same point.

10. Hence it follows, that the six triangles are equilateral triangles.

The following are questions, which the master may seasonably and appropriately ask his pupils, relative to polygons, in general:—

M.—What is the sum of the interior angles of a heptagon—of an octagon? In what manner may the sum of the interior angles of any polygon be determined?

P.—Every polygon may be resolved into as many triangles as the figure has sides; the sum total of their angles is, therefore, equal to twice as many right angles as the figure has sides.

Now, the sum of the angles about the common vertex of these triangles = 4 rt. ∠s; therefore, the sum or the interior angles of any polygon is equal to twice as many right angles as the figure has sides less [*minus*] four.

M.—If the number of sides be three, four, five, six, seven, &c., what is the sum of the interior angles o. figures so constituted, respectively?

P.—Two, four, six, eight, ten, &c. right angles.

M.—What relation have the numbers three, four, five, six, seven, &c. to each other?

P.—They form an *arithmetical* series; they are the natural numbers, beginning from *three,* in an *ascend*ing order.

M.—And, what relation have the numbers two, four, six, eight, ten, &c. to each other?

P.—They form, likewise, an arithmetical series.

M.—Endeavour to express, in words, the relation which the sides and sum of the interior angles of polygons have to each other, when taken in *such* an order.

P.—The sides form an arithmetical series whose common difference is *one,* and first term *three;* and the respective sums of the interior angles form an arithmetical series whose common ratio and first term is *two.*

M.—In a triangle, when each side is produced, what is the sum of the three exterior angles?

P.—Four right angles.

(*Demonstrated before.*)

M.—Produce each of the sides of a quadrilateral figure,—a pentagon,—a hexagon—of any polygon; and find the sum of the exterior angles.

The pupils will find this sum to be, in every case, equal to four right angles.

Hence, if each of the sides of *any* rectilineal figure be produced, the sum of the exterior angles is equal to four right angles.

CHAPTER II.

SECTION I.

ONE CIRCLE—ONE AND TWO STRAIGHT LINES IN A CIRCLE.

M.—State all you know of a circle.

Let the pupils state what they remember of Lesson XI. [Introd.]; and, to facilitate their recollection, let the sphere, cylinder, and cone, be presented to them.

M.—By what means may a correct circle be described?

P.—By a pair of compasses, or by means of a string.

The master has, here, a fit opportunity of making several observations on the use of the Mathematician's Compass in describing a circle, and of instructing the pupils to draw circles without its assistance.

M.—What is the *fixed* point called?

P.—The centre of the circle.

M.—And, what is the *curved* line called?

P.—The circumference of the circle.

M.—And, what is the space bounded by the circumference called?

P.—The *area* of the circle, or the circle.

M.—What do you know respecting the position of the centre of a circle with regard to the circumference?

P.—The centre of a circle is such, that all straight lines drawn from it to the circumference are *equal* to each other.

M.—What are such lines called?

P.—*Radii* of the circle; and one is called a *Radius*.

M.—Can a circle have more than one centre?

P.—No; for, then, there would be radii *not* equal to one another.

M.—And, what is a straight line passing through the centre and terminated both ways by the circumference called?

P.—A *diameter* of the circle.

M.—Compare the portions of a circle obtained by drawing a diameter.

P.—They are equal to one another, and are called *semi*-circles [*half*-circles].

M.—Compare, likewise, the portions of the circumference cut-off by a diameter.

P.—They are, likewise, equal to one another.

M.—These and other portions of the circumference are called *arcs*. When is an arc a *semi*-circumference?

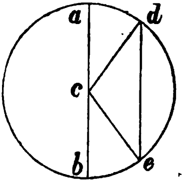
P.—When the straight line joining its extremities is a diameter.

M.—Describe a circle, and draw, in it, any straight line terminated both ways by the circumference. In how many different ways can such a line be drawn?

P.—Either passing through the centre, or not.

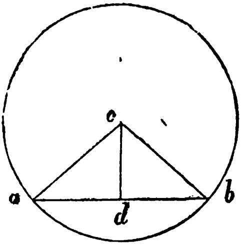
M.—In a circle, a straight line, which is terminated both ways by the circumference, and which is not a diameter, is called a *Chord.* Compare a chord with a diameter.

P.—A chord is always *less* than a diameter.



For, join cd and ce;
then ∵ any two sides of a triangle are together greater than the third side,
$\therefore cd + ce > de$.
But $cd + ce = ab$, the diameter of the circle;
$\therefore ab > de$.

M.—From the centre erect a perpendicular on a chord; and, then, compare the segments of the chord.



P.—The chord ab is bisected by the perpendicular cd.

For, join ac, cb; $\therefore$ $\triangle\, acb$ is an isosceles $\triangle$;
and, cd being perpendicular to ab,

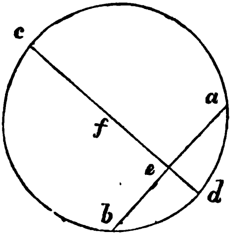
$$ad = bd.$$

M.—Hence, if, from the centre, cd be drawn, bisecting the chord ab, what are the angles which it makes with ab?

P.—Right angles: for, then, $ca = cb$, and $ad = db$, and cd is common; ∴ $\triangle acd = \triangle cdb$, and ∴ $\angle cda = \angle cdb$.

M.—I think, from the truth you have here demonstrated, you will be able to devise some method of *finding* the centre of a circle.

P.—Yes; for, it must be in the perpendicular which is drawn bisecting a chord; thus:



draw any chord ab in a circle,
and let cd bisect ab, and be perpendicular to it;
bisect cd in f: then f must be the centre of the circle.

M.—Find the different ways in which two straight lines may be drawn in a circle.

[The pupils ascertain this.] The most important of the several cases are,

1. When the two straight lines are diameters.

M.—If two straight lines in a circle are diameters, where is their point of intersection?

P.—In the centre of the circle.

M.—How may two diameters intersect each other?

P.—Either at right angles, or not.

M.—Compare the *arcs* intercepted by these diameters.

P.—They are equal to one another; and each is *one-fourth* part of the circumference.

M.—Compare the *areas* intercepted by the diameters and arcs.

P.—They are equal to one another; and each is *one-fourth* part of the circle.

M.—They are, on this account, called *Quadrants.* [fr. *quadrans,* Lat.] If, then, an angle at the centre of a circle is a right angle, what part of the circumference is the arc which subtends it?

P.—*One-fourth.*

M.—And, if the arc is *not* a fourth part of the circumference, what must the angle which it subtends at the centre be?

P.—Either an *obtuse* or an *acute* angle, according as the arc, which it subtends, is *more* or less than one-fourth of the circumference.

M.—Compare the arcs and the areas intercepted by two diameters which do *not* intersect each other at right angles.

P.—The opposite arcs and areas are equal to each other; the smaller arcs subtend the equal acute angles, and the two greater arcs subtend the equal obtuse angles at the centre.

M.—And, if the acute angles be made greater or smaller, what will, consequently, be the case with the subtending arcs?

P.—They will, consequently, become greater or smaller also.

M.—Hence, what relation exists between the arcs of a circle and the angles which they subtend at the centre?

P.—They increase or decrease according as the angles at the centre increase or decrease.

M.—It is from this circumstance, that arcs are said to be the *measures* of the angles which they subtend at the centre.

2. When two straight lines in a circle are *not* diameters.

M.—When two straight lines in a circle are not to be diameters, in what two different ways can they be drawn?

P.—Intersecting one another, or, not intersecting one another.

M.—If they do *not* intersect each other, when are they equal to each other?

P.—When they are equi-distant from the centre.

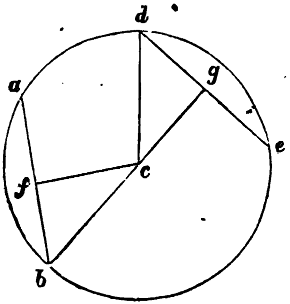
M.—By what means would you measure the distance of a straight line from a point?

P.—By a perpendicular drawn from the point to the line.

M.—Why a *perpendicular?* Could any other straight line serve the same purpose?

P.—No: for, from the point, any number of *unequal* straight lines may be drawn to a straight line, but only one perpendicular.

M.—Then, demonstrate that chords which are equidistant from the centre are equal to each other.



P.—From the centre c, draw the perpendiculars $c f$, $c g$; and join $c b$, $c d$.

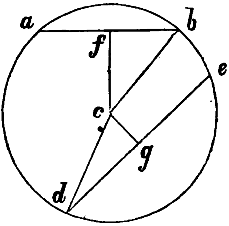
Then ∵ $a b$, $d e$, are equi-distant from the centre,
$c f = c g$;
and $c b = c d$, as they are radii;
and, $b f$ is one-half of $a b$;
also, $g d$ is one-half of $d e$.
But, ∵ $\angle\ c f b$ is a rt. $\angle$;
$\therefore c b^2 = c f^2 + b f^2$;
similarly, $c d^2 = g d^2 + c g^2$.
But, $c b^2 = c d^2$ (because $c b = c d$);
$\therefore c f^2 + b f^2 = g d^2 + c g^2$.
But, $c f^2 = c g^2$ (because $c f = c g$);
$\therefore b f^2 = g d^2$,
and, $\therefore b f = g d$.
But $b f$ is one-half of $a b$,
and $g d$ is one-half of $d e$;
$\therefore a b = d e$.

M.—And, when it is known that chords in a circle are equal to each other, what inference must be drawn respecting their *distances* from the centre?

P.—Their distances must be equal:
for, $\because ab = de$, and cf, cg are perpendiculars, drawn to them from the centre,
$\therefore bf = gd$,
and $cb = cd$;
$\therefore cf^2 + bf^2 = cg^2 + gd^2$ (as before).
But $bf^2 = gd^2$;
$\therefore cf^2 = cg^2$,
and $\therefore cf = cg$;
that is, the perpendiculars are equal; and, therefore, *the chords a b, d e, are equi-distant from the centre.*

M.—And, when are two chords *not* equal to each other?

P.—When they are at *unequal* distances from the centre.



For, if de be $> ab$,
dg is $> bf$.
But, $cf^2 + bf^2 = cg^2 + dg^2$,
and $dg^2 > bf^2$;
$\therefore cf^2 > cg^2$;
that is, $cf > cg$.

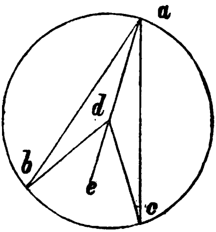
Hence, the chords are at *unequal* distances from the centre: and the lesser chord, *a b*, is farther from the centre than the greater chord, *d e*.

3. When two straight lines in a circle *intersect* each other.

M.—If two chords in a circle intersect each other, where may their point of intersection be?

P.—Either in the *circumference* of the circle, or *within* the circle.

M.—In the former case, the angle made by two chords meeting in the circumference is called the angle *at* the circumference—



[in contra-distinction to the angle *b d c* at the centre]. What have the angles *b a c* and *b d c in common?*

P.—The arc *b c* subtending them both.

M.—Hence, these angles are said to *stand upon the same arc.* Compare them.

P.—The angle *b d c* at the centre is greater than the angle *b a c* at the circumference;—it is *double* the angle at the circumference.

For, join *a d* and produce it to *e;*

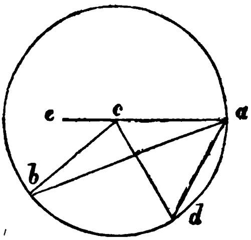
then ∵ $da = db$,

$\angle dab = \angle dba$;

and ∴ the ext. $\angle edb$ is double $\angle dab$.

For a similar reason, $\angle edc$ is double $\angle dae$;

and ∴ the whole $\angle bdc$ is double $\angle bac$.



M.—In this figure, the angle *b a d* is an angle at the circumference, and the angle *b c d*, at the centre; and they stand upon the same arc *b d*. Is ∠ *b c d*, likewise, *double* ∠ *b a d?*

P.—Yes. Join *a c* and produce it to *e;*

∴ ∠ *e c d* is double ∠ *c a d;*
also, ∠ *e c b* is double ∠ *c a b;* } as previously.
∴ remaining ∠ *b c d* is double the remaining ∠ *b a d.*

M.—Hence, when an angle at the circumference is one-half of a right angle, what is the angle at the centre, upon the same arc?

P.—Double one-half of a right angle,—that is, one right angle.

M.—And what part, therefore, of the whole circumference is the arc upon which they stand?

P.—One-fourth of the circumference.

M.—And, when an angle at the circumference is a right angle, what is the corresponding angle at the centre?

P.—There can, in this case, be *no* angle at the centre; because, no angle can be equal to *double a right angle*, or, to two right angles.

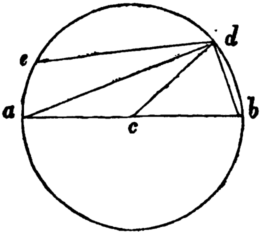
M.—Now, what part of the circumference is the arc which subtends a right angle at the circumference?

P.—It must be a semi-circumference.

M.—If, then, an angle at the circumference stands upon a semi-circumference, what must that angle be?

P.—A right angle.

M.—Demonstrate that—When an angle at the circumference stands upon a semi-circumference, it is a right angle.



P.—If the arc aeb is a semi-circumference, ab passes through the centre c. Join cd:

then $\because cd = ca$, and $cd = cb$,

$\angle cda = \angle cad$, and $\angle cdb = \angle cbd$;

and $\therefore$ whole $\angle adb = \angle$s $cad + cbd$.

Now, if one of the angles of a triangle is equal to the sum of the other two, it is a right angle;

$\therefore \angle adb$ is a rt. $\angle$.

M.—And, if it be known that an arc is less than a semi-circumference, what must be concluded with respect to the angle which it subtends at the circumference?

P.—It must be less than a right angle; it must be an *acute* angle.

M.—Demonstrate this.

P.—∵ the arc $a\,d\,b$ is a semi-circumference,
∴ the arc $d\,b$ is less than a semi-circumference.
But, in $\triangle\ a\,d\,b$, $\angle\ a\,d\,b$ is a rt. $\angle$;
∴ $\angle\ d\,a\,b$ is *less* than a rt. $\angle$.

M.—And, if the arc be greater than a semi-circumference—?

P.—The angle which it subtends at the circumference must be greater than a right angle; it must be *obtuse*.
For, ∵ the arc $a\,b$ is a semi-circumference,
∴ the arc $b\,a\,e$ is greater than a semi-circumference.
But $\angle\ a\,d\,b$ is a rt. $\angle$;
∴ $\angle\ e\,d\,b >$ rt. $\angle$,
i. e. $\angle\ e\,d\,b$ is an obtuse angle.

M.—There is an important truth dependent on the relation of angles at the centre and angles at the circumference, upon the same arc. At the centre, can there be several angles which stand upon the same arc?

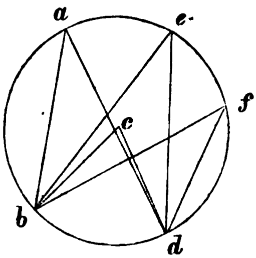
P.—No.

M.—But, can the same arc subtend several angles at the circumference?

P.—Yes,—an indefinite number.

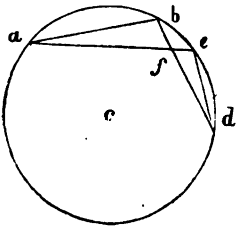
M.—And, what may be concluded respecting them?

P.—They must all be *equal*.



Join *c b*, *c d*;
∴ ∠ *b c d*, at the centre *c*,
is double of each of ∠s *b a d*, *b e d*, and *b f d*;
and ∴ ∠ *b a d* = ∠ *b e d* = *b f d*.

M.—Look at the figure you have adopted, in demonstrating this truth. What part of the circumference is the arc *b d*?—what kind of angles are *b a d*, *b e d*, *b f d*?—Now, show that angles which stand on arcs greater than the semi-circumference are, likewise, equal to each other.



P. ∵ arc *a d* > semi-circumference,
∴ arc *a b e d* < semi-circumference;
and ∴ ∠ *b a e* = ∠ *b d e* (as before).

But $\angle\, afb = \angle\, dfe$, inasmuch as they are vertical $\angle$s;

$\therefore$ in $\triangle$s abf, dfe, there are 2 $\angle$s of the one $=$ 2 $\angle$s of the other, each to each;

$\therefore$ rem. $abd =$ rem. $\angle\, aed$.

M.—We have now arrived at the last case,—"when the point of intersection of two chords falls *within* the circle." If the point of intersection is in the centre, what, then, are the chords?

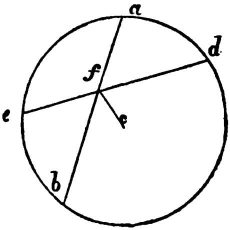
P.—Diameters of the circle.

M.—And, if we conceive rectangles formed by their segments, what conclusion must necessarily follow respecting them?

P.—Such rectangles are squares, and are equal to each other.

M.—But, if the point of intersection be *not* the centre, are the segments, then, equal to each other?

P.—No.



Join cf:

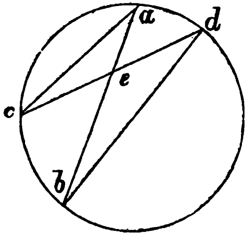
then, if $bf = fa$,

$\angle\, cfb$ is a rt. $\angle$;

and if $ef = fd$,

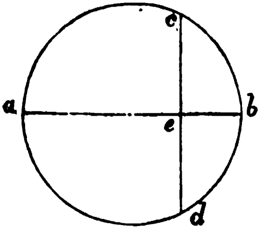
$\angle\, cfe$ is likewise a rt. $\angle$;
and $\therefore \angle\, cfb = \angle\, cfe$,
which is impossible :
$\therefore$ ab and ed do *not* bisect each other.

M.—Again, if the point of intersection is *not* the centre of the circle, are, then, the rectangles contained by their segments equal to each other?



P.—Yes: the rectangle contained by $a e$, $e b$, is equal to the rectangle contained by $c e$, $e d$.
Join $a c$ and $d b$:
$\therefore \angle\, acd = \angle\, abd$, as they are on the same arc $a d$;
similarly, $\angle\, cab = \angle\, cdb$,
and $\angle\, cea = \angle\, bed$, as they are vertical $\angle$s:
$\therefore \triangle\, aec$ is similar to $\triangle\, bed$;
and $\therefore ae : ed :: ec : eb$;
and, hence, $ae \times eb = ed \times ec$,
that is, rectangle $a e$, $e b$ = rectangle $e d$, $e c$.

M.—There is a particular case depending on this truth.—Let one of the two straight lines be a diameter, and the other a chord at right angles to it: find how the truth, you have demonstrated, is then modified.



P.—The rectangle *a e*, *e b* = rectangle *c e*, *e d*.
But *c e* = *e d*, because *a b* is at right angles to *c d*;
∴ rectangle *c e*, *e d* = ce^2 or de^2;
and ∴ rectangle *a e*, *e b* = ce^2.

M.—Express this truth in words.

P.—If, from a point in the circumference of a circle, a perpendicular be drawn to the diameter, the rectangle contained by the segments of the diameter is equal to the square of the perpendicular.

SUBSTANCE OF SECTION I.

1. A circle has only one centre.
2. A diameter is the longest straight line that can be drawn in a circle.
3. An arc of a circle is a portion of its circumference.
4. The straight line joining the extremities of an arc is called a chord.
5. A perpendicular drawn from the centre to a chord bisects the chord.
6. If a chord be bisected, the straight line joining the point of bisection and the centre of the circle is perpendicular to the chord.

7. Arcs are the *measures* of the angles which they subtend at the centre.

8. Chords which are equi-distant from the centre of a circle are equal to each other.

9. Equal chords, in a circle, are equi-distant from the centre.

10. Chords, in a circle, which are not equi-distant from the centre are not equal to each other, and the lesser chord is farther from the centre than the greater chord.

11. The angle at the centre of a circle is double the angle at the circumference, upon the same arc.

12. At the circumference, an angle which stands upon a semi-circumference is a right angle.

13. At the circumference, an angle which stands upon an arc less than a semi-circumference is an acute angle; and that which stands upon an arc greater than a semi-circumference is an obtuse angle.

14. At the circumference, angles which stand upon the same arc are equal to each other.

15. In a circle, if two chords intersect each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

16. If, from any point in the circumference, a perpendicular be drawn to a diameter, the rectangle contained by the segments of the diameter is equal to the square of the perpendicular.

SECTION II.

THREE AND MORE *straight lines* IN A CIRCLE.

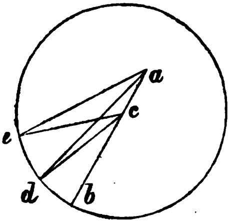
M.—If, from any point in a circle, three straight lines be drawn to the circumference, what will result from comparing them ?

P.—Nothing definite can be said of them, except when the point is the centre.

M.—But, if one of the three straight lines passes through the centre—?

P.—The straight line which passes through the centre is the greatest of the three.

Let $a\,b$ pass through the centre c; $a\,b$ shall be greater than $a\,d$ or $a\,e$.



Join $c\,d$, $c\,e$:

then, $\because$ c is the centre of the circle,

$c\,b = c\,d = c\,e$.

But $a\,c + c\,d > a\,d$;

$\therefore a\,c + c\,b > a\,d$;

that is, $a\,b > a\,d$.

Similarly, $a\,b > a\,e$;

$\therefore$ $a\,b$ is the greatest straight line.

[In such cases, of course, the term "greatest" refers to *length* solely.]

M.—Now, compare the other two lines, *a d* and *a e*.

P. *a d* is greater than *a e*;

$\because$ *a c* and *c d* = *a c* and *c e*, each to each.

But $\angle\, a c d > \angle\, a c e$;

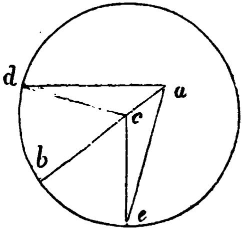
$\therefore a d > a e$.

M.—Express this in words.

P.—Of three straight lines that can be drawn from any point in a circle to the circumference, whereof one passes through the centre, the greatest is that through the centre; and the one that is nearer to that which passes through the centre is greater than the one more remote.

M.—But, can there be drawn two *equal* straight lines from the point *a* to the circumference?

P.—Yes: one on each side of *a b*,—that is, one on each side of the greatest line.



Make $\angle\, d c b = \angle\, e c b$,
and join *a d* and *a e*:
$\therefore$ *a c*, *c d* = *a c*, *c e*, each to each,
and $\angle\, a c d = \angle\, a c e$;
$\therefore$ base *a d* = base *a e*.

M.—Can there be drawn, from the point to the circumference, a third line which shall be equal to *a d* or *a e?*

P.—No; because it has been proved, that any line which is nearer to *a b,* is greater than one that is more remote: and, therefore, any line drawn on either side of *a b* must be less or greater than *a d* or *a e.*

M.—How many equal straight lines can be drawn from the point *a* to the circumference?

P.—Only two single lines; but as many *pairs* of equal lines as you please.

M.—Let us suppose, then, that three lines, from a certain point to the circumference, are equal to each other,—where must that point be in the circle?

P.—It must be the centre of the circle.

M.—Then, how many equal lines, at least, are required to determine whether a certain point is the centre of a circle?

P.—At least, *three* equal straight lines to the circumference.

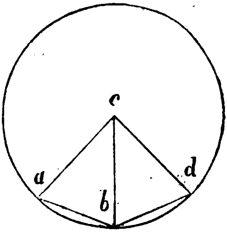
M.—Now, draw three straight lines, in all possible ways, in a circle; and find what results arise from the investigation of each case.

P.—1. Three straight lines in a circle may all be diameters, or all radii.

If they are diameters, no particular truth arises from the circumstance.

If they are radii, the angles which they make at the centre may be equal to each other, or they may not.

If the angles at the centre are equal, the arcs which subtend them must be equal, (since arcs are the measures of angles at the centre,) and the straight lines which join the extremities of the radii are equal.



For, $\triangle\, c\,a\,b$ evidently $= \triangle\, c\,b\,d$;
and $\therefore a\,b = b\,d$.

M.—Hence, chords which cut-off equal arcs of a circle must be——?

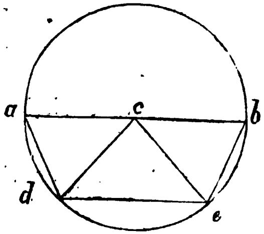
P.—Equal to each other.

M.—And, equal chords in a circle must cut-off——?

P.—Equal arcs.

But, if the angles at the centre are *not* equal to each other, the corresponding arcs are not equal to each other; nor are the lines which join their extremities equal to each other and the greater line is that which subtends the greater angle.

2. Of three straight lines in a circle, two may be radii, and the third a diameter.



If $\angle\, acd = \angle\, dce = \angle\, ecb$,
each of them $= \frac{2}{3}$s of a rt. $\angle$;
$\therefore \angle$s $cad + cda = \frac{4}{3}$ rt. $\angle$.
But $\angle\, cad = \angle\, cda$;
$\therefore$ each of $\angle$s cad, $cda = \frac{2}{3}$ rt. $\angle$:
hence, $\triangle\, acd$ is equiangular,
and $\therefore$, also, equilateral,
and $ca = cd = ad$.
But $\triangle\, acd = \triangle\, dce = \triangle\, ecb$;
$\therefore ad = de = eb$.

3. Of three straight lines in a circle, two may be diameters, and the third a radius.

No particular truth arises from this case.

4. They may be chords.

These, if equal to each other, are equi-distant from the centre.

And, if equi-distant from the centre, they are equal to each other.

Two of them may be equal to each other; the third, then, is not equi-distant, with them, from the centre.

They may be all parallel, or two only may be parallel with each other.

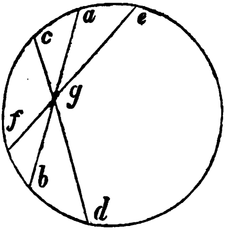
No particular truth arises from this case, unless a perpendicular be drawn, from the centre, upon one of them; it is, likewise, perpendicular to each of them, and bisects each of them.

5. Three straight lines in a circle may intersect each other.

Their point of intersection may be within the circle or in the circumference.

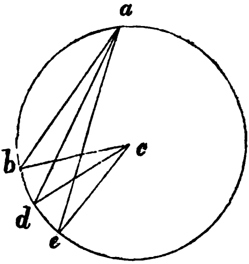
If it is within the circle, and the lines are equal to each other, the point of intersection is the centre. (This has already been shown.)

If the lines are not equal to each other, the rectangles contained by the segments of each are equal to each other.



For, the rectangle ag, gb = rectangle cg, gd;
also, rectangle ag, gb = rectangle eg, gf:
∴ rectangle ag, gb = rectangle cg, gd = rectangle eg, gf.

If they are in the circumference, and the angles which they make are equal to each other, the arcs which subtend them are, likewise, equal to each other.



For, let $\angle bad = \angle dae$:

$\therefore \angle bcd = \angle dce$,—as $\angle$s at the centre;
and $\therefore$ arc bd = arc de.

6. They may intersect each other in two points.

Nothing determinate can be said of them, in this case, unless more is known.

7. They may intersect in three points, and form a triangle.

That triangle may be right-angled, obtuse-angled, or acute-angled.

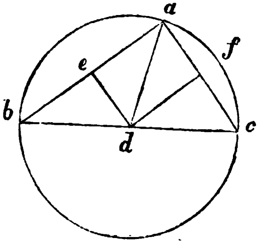
M.—Where are the angular extremities of such a triangle?

P.—In the circumference.

M.—A triangle, or any other figure, of which all the angles are in the circumference of a circle, is said to be *inscribed* in the circle. Where is the centre of the circle, when the inscribed triangle is right-angled?

P.—In the intermediate or middle point of the side which subtends the right angle; because every right angle at the circumference is subtended by a semi-circumference, and the line which joins the extremities of a semi-circumference is a diameter.

M.—Endeavour to demonstrate this, by means of auxiliary lines.



P.—Let $\angle\, b\,a\,c$ be a rt. $\angle$;
bisect $b\,c$ in d:
the point d shall be the centre of the circle.
Draw $d\,e$, $d\,f$ at rt. $\angle$s to $a\,b$, $a\,c$, respectively,
and join $d\,a$:
then $\because d\,e$ is at rt. $\angle$s to $a\,b$,
$\therefore a\,e = b\,e$,
and $d\,e$ is common to $\triangle$s $d\,e\,b$, $d\,e\,a$;
$\therefore d\,b = d\,a$.
In the same way, it may be shown that
$d\,a = d\,c$;
$\therefore d\,b = d\,a = d\,c$;
and $\therefore d$ is the centre of the circle.

M.—But, when the inscribed triangle is obtuse or acute, where is the centre?

P.—When it is obtuse, the centre of the circle must be *without* the triangle, beyond the side which subtends the obtuse angle; because an obtuse angle is subtended by an arc which is greater than a semi-circumference.

If it is an acute-angled triangle, the centre of the circle is within the triangle.

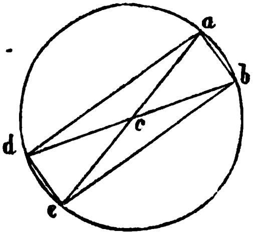
M.—Yes: and, when the inscribed triangle is equilateral, where is the centre of the circle?

P.—In the centre of the triangle; because, since the sides are equal, they are equi-distant from the centre.

The pupils may, now, be left to draw *four* straight lines in a circle. The truths, arising from their various combinations, will afford the means of recapitulating most of those positions which have before been established, and which need, here, no repetition. Those which are of chief importance, however, are the following.

M.—If *four* chords be drawn so as to join the extremities of two diameters, what may be said of them.

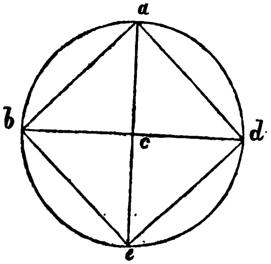
P.—The opposite chords are equal to each other; and, therefore, they form a parallelogram.



For, ca, $cb = cd$, ce, each to each,
and $\angle acb = \angle dce$;
$\therefore ab = de$;
similarly, $ad = be$.
But, if the opposite sides of a quadrilateral figure are equal to each other, the figure is a parallelogram;
$\therefore abed$ is a parallelogram.

M.—And, when will the inscribed quadrilateral figure be a square?

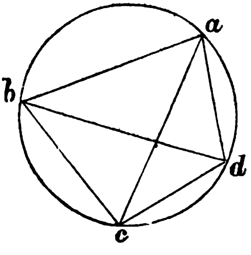
P.—When the diameters cut each other at right angles.



For, then $ab = bd$;
and ∴ *abed* is equilateral.
But ∠ *bad* is a rt. ∠ [because it is subtended by the semi-circumference *bed*];
∴ *abde* is rectangular,
and ∴ *aebd* is a square.

M.—Draw any four chords in a circle, so as to form a quadrilateral figure, and determine what may be said of the sum of the opposite angles.

P.—The sum of the opposite angles of any quadrilateral figure, inscribed in a circle, is equal to two right angles.



Join ac and bd:
then $\because$ $\angle$s cad, cbd are upon the same arc cd,
$\angle cad = \angle cbd$;
for a similar reason, $\angle bac = \angle bdc$;
$\therefore$ whole $\angle bad = \angle$s $cbd + bdc$.
To each of these equals add $\angle bcd$;
$\therefore$ $\angle$s $bad + bcd = \angle$s $cbd + bdc + bcd$.
But $\angle$s $cbd + bdc + bcd = 2$ rt. $\angle$s;
$\therefore$ $\angle$s $bad + bcd = 2$ rt. $\angle$s;
and, $\because$ all the angles of the figure $abcd = 4$ rt. $\angle$s,
$\therefore$ $\angle$s $abc + adc = 2$ rt. $\angle$s.

M.—Then, what are the *only* quadrilateral figures which may be inscribed in a circle?

P.—A square; a rectangle; and a trapezium, whose opposite angles are together equal to two right angles.

The pupils are, now, required to draw *five, six, seven, &c. chords* in a circle. These combinations, however, offer no important results which can be adequately appreciated by them, at this stage of their geometrical knowledge. From this general remark may be excepted, however, the particular case of an inscribed equilateral and equiangular hexagon,—each side of which is equal to the radius of the circle; whence the truth, that, *the radius of a circle may be applied six times, exactly, to the circumference.*—The demonstration of this being easy, will be readily discovered by the pupils.

SUBSTANCE OF SECTION II.

1. If, from a point in a circle, there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.

2. Of all straight lines that can be drawn from a point in a circle to the circumference, the greatest is that which passes through the centre.

3. And, of the others, that which is nearer to the one in which the centre is, is greater than any one more remote.

4. And, from the same point, only *pairs* of *equal* straight lines can be drawn to the circumference.

5. Chords which cut-off equal arcs are equal to each other.

6. If chords are parallel, a perpendicular drawn from the centre to one of them, when produced to their circumference, is perpendicular to each of them, and it bisects them.

7. When angles at the circumference are equal to each other, the arcs upon which they stand are, likewise, equal to each other.

8. A figure is said to be *inscribed* in a circle, when all its angular extremities are in the circumference of the circle.

9. If a right-angled triangle be inscribed in a circle, the intermediate or middle point of the side which subtends the right angle is the centre of the circle.

10. If an obtuse-angled triangle be inscribed in a circle, the centre of the circle is without the

triangle, beyond the side which subtends the obtuse angle.

11. When an acute-angled triangle is inscribed in a circle, the centre of the circle is within the triangle.

12. When an equilateral triangle is inscribed in a circle, the centre of the circle is, likewise, the centre of the triangle.

13. The four straight lines which join the extremities of two diameters, in a circle, form a rectangular parallelogram.

14. If two diameters intersect each other at right angles, the four straight lines joining their extremities form a square.

15. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

16. A square,—a rectangle,—and a trapezium of which the opposite angles are together equal to two right angles,—are the *only* quadrilateral figures which can be inscribed in a circle.

17. If an equilateral and equiangular hexagon is inscribed in a circle, each of its sides is equal to the radius of the circle.

18. Hence, the radius of any circle may be applied *six* times to the circumference.

SECTION III.

LINES *without* A CIRCLE—TANGENTS.

The object of this section is the investigation of such truths as arise from the relations of straight lines *without* a circle. The method of proceeding is here, likewise, closely similar to that which has been observed in the preceding sections.

A point is, first, assumed without the circle, from which straight lines are drawn so as to intersect the circle.

Next,—one tangent is drawn, and the angles, which a diameter, drawn to the point of contact, makes with it are investigated.

Two tangents, *three* tangents, &c. are, then, drawn in succession, and the nature of the circumscribing figures is investigated.

M.—When is a point *without* a circle?

P.—When its distance from the centre is greater than that measured by a radius.

M.—If, from a point without a circle, it be required to draw a straight line to the circle,—find in how many different ways this can be done.

P.—A straight line, so drawn, may *cut* the circle;
or, it may only *meet* the circle;
or, it may *touch* the circle;
or, it may neither cut it, meet it, nor touch it.

M.—If a straight line is drawn so as to cut a circle,—determine in what manner it may cut it.

P.—It may either pass through the centre, or not pass through the centre.

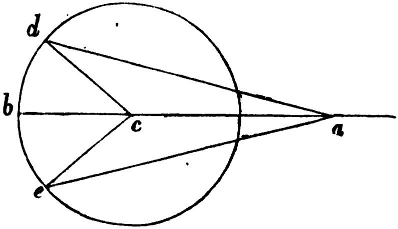
M.—If, from the same point, two lines be drawn so as to cut the circle, in what manner may they cut it?

P.—One of them may pass through the centre, and the other may not pass through it; or, neither of them may pass through it.

M.—Compare two such lines, in each case.

P.—If one of them passes through the centre, it is greater than the one which does not pass through the centre.

Let ab pass through the centre c, and ad cut the circle; ab shall be greater than ad.



Join dc;

$\therefore dc + ca > ad$:

but $dc = cb$;

$\therefore ab > ad$.

And, if two straight lines be drawn from the point a, they may be equal to each other, or they may not be equal to each other;

ad is equal to ae,

when $\angle dcb = \angle ecb$;

∵ then, $\angle\, d c a = \angle\, e c a$,
and $d c$, $c a = e c$, $c a$, each to each;
∴ $a d = a e$.

M.—Hence, of all straight lines which can be drawn from the point *a* so as to cut the circle, what will be the greatest?

P.—The one which passes through the centre; and of the others, that which is nearer to the one in which is the centre, will be greater than one more remote. Also, only two equal straight lines can be drawn from the same point so as to cut the circle, one on each side of the greatest line.

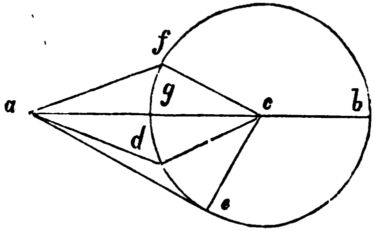
Obs.—The demonstration of these positions being similar to the analogous case in the preceding section, it is unnecessary to repeat it here. The *pupils*, however, must demonstrate what they have stated above, —and, indeed, should *always* be required to demonstrate whatever admits of demonstration.

M.—When do you consider a straight line as meeting, and when as touching, a circle?

P.—A straight line *touches* a circle when it meets the circumference, and, being produced, does not cut t.

M.—Compare straight lines drawn, from the same point, to *meet* the circle.

P.—That is the least line which, when produced, passes through the centre of the circle; and of the others, that which is *farther* from the *least* line is always greater than one more contiguous to it.



Of the straight lines ag, ae, let ag, produced, pass through the centre c, and

join cd, ed:

then $\because cd + da > ca$,

and $cd = cg$,

$\therefore$ remainder $ag < ad$;

that is, the one which, when produced, passes through the centre, is less than one which, when produced, does not pass through the centre:

$\therefore ag$ is the least.

Again, $\because d$ is a point in $\triangle aec$,

$\therefore ae + ec > ad + dc$;

but $ec = dc$;

$\therefore ae > ad$;

that is, a line which is farther from the least line is greater than one nearer to it.

Also, if $\angle dca = \angle fca$,

$ad = af$;

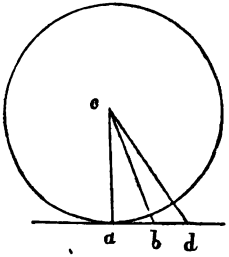
$\because \triangle adc = \triangle afc$.

Hence, only a pair of equal straight lines can be drawn from the same point so as to meet the circle, one on side of the least line.

M.—When a straight line touches a circle, it is called a *tangent;* and the point in which it touches the circumference is called the *point of contact.*

From the point of contact draw a straight line so as to cut the circle, and determine the angles it makes with a tangent.

P.—A straight line from the point of contact either passes through the centre of the circle, or it does not pass through the centre. If it passes through the centre, it is at right angles to the tangent.



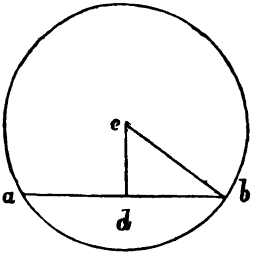
For, if *a* is the point of contact, any straight lines, *c b*, *c d*, from the centre are always greater than *c a:* ∴ *c a* is the least line which can be drawn from a point to a straight line;

∴ *c a* is at rt. ∠s to *a b*.

M.—But, could not another point, *b* for instance, be likewise in the circumference of the circle?

P.—No: for, then, *a b* would cut the circle, and, therefore, not be a tangent.

M.—Can you show, that a straight line joining any two points in the circumference of a circle is *within* the circle?



Let *a*, *b*, be any two points in the circumference: the straight line joining them is ***within*** the circle.

From *c*, the centre, draw *c d* at rt. ∠s to *a b*, and join *c b*;

∴ *c b* > *c d* (because it subtends a rt. ∠),
and ∴ the point *d* is within the circle:
but the point *d* is in the straight line *a b*;
∴ the points *a*, *b*, are in the same line with *d*,
and ∴ *a b* is ***within*** the circle.

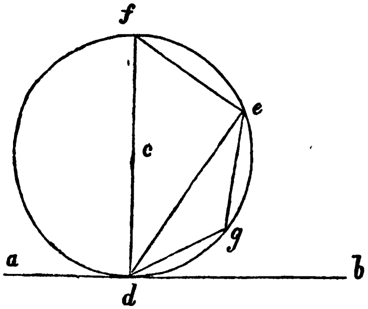
M.—Hence, what may be said of a tangent?

P.—A tangent touches a circle in one point only; and the straight line from the centre to the point of contact is at right angles to the tangent.

M.—If, then, the angle which a straight line from the centre makes with another line meeting the circle is ***not*** a right angle, what must be concluded?

P.—That the straight line which meets the circle is ***not*** a tangent, and that, being produced, it must, therefore, cut the circle.

M.—Now, investigate the angles which a straight line from the point of contact makes with the tangent, when it does ***not*** pass through the centre.



Let ab be a tangent,

d the point of contact,

and de a straight line, *not* passing through the centre c:

$\angle edb = \angle$ upon the arc dge,

and $\angle eda = \angle$ upon the arc dfe.

Draw dcf, and join fe;

in the arc de take any point, g,

and join dg, ge;

$\therefore \angle fdb$ is a rt. $\angle$.

But $\angle def$ is a rt. $\angle$, it being $\angle$ upon the semi-circumference fd;

$\therefore \angle fdb = \angle def$:

and $\angle def = \angle$s $efd + edf$;

$\therefore \angle fdb = \angle$s $efd + edf$,

$\therefore \angle$s $fde + edb = \angle$s $efd + edf$,

and $\therefore \angle edb = \angle efd$, upon the arc dge.

Again, $\because$ opposite $\angle$s $efd + dge = 2$ rt. $\angle$s,

$\therefore \angle$s $eda + edb = \angle$s $efd + dge$:

but, $\angle edb = \angle efd$;

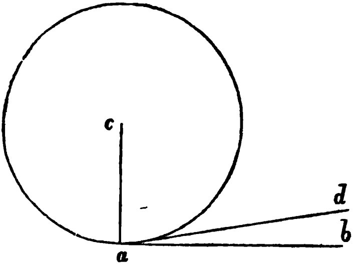
$\therefore \angle eda = \angle dge$, upon the arc dfe.

M.—Express this truth in words.

P—If, from the point of contact, a straight line be drawn so as to cut the circle, the angles which this line makes with the tangent are equal to the angles upon the adjacent arcs.

M.—You said that, a straight line from the point of contact either passes through the centre or it does not; but, cannot a straight line be drawn between the tangent and the circumference?

P.—No.



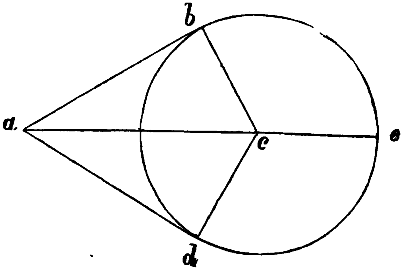
For, $\angle$ *c a b* is a rt. $\angle$;
$\therefore$ $\angle$ *c a d* is not a rt. $\angle$;
and $\therefore$ *a d* is *not* a tangent.
Hence, *a d* must cut the circumference.

M.—Consequently, how *many* tangents can be drawn from the same point *without* a circle?

P.—Only two tangents, one upon each side of the straight line which passes, from the point, through the centre.

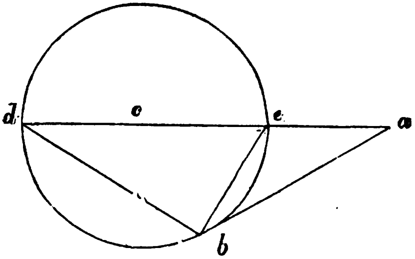
M.—Draw these, and compare them.

P—They are equal to each other.



Join $c b$, $c d$:
$\therefore \angle$s $c b a$, $c d a$ are rt. $\angle$s,
and $\therefore c a^2 = c b^2 + a b^2 = c d^2 + a d^2$;
$\therefore a b^2 = a d^2$, and $\therefore a b = a d$.

M.—There is another interesting truth respecting a tangent, connected with those you have already ascertained. I will assist you in discovering it.



Let $a b$ be a tangent, and
$a d$ a straight line cutting the circle in e;
join $d b$ and $e b$.
Compare the $\triangle$s $a d b$, $a e b$.

P.—$\triangle\, adb$ is similar to $\triangle\, aeb$.
For, $\because\, be$ cuts the circle,
$\therefore\, \angle\, abe = \angle\, adb$, upon the arc be;
and $\angle\, bad$ is common to the two $\triangle$s:
$\therefore\, \angle bea = \angle\, abd$;
$\therefore\, \triangle\, aeb$ is *similar* to $\triangle\, abd$.

M.—And, what are the homologous sides, in these triangles?

P.—ae, ab, and eb, are homologous to ab, ad, and db, respectively.

M.—What proportion results from the first pair of these homologous sides?

P. $ae : ab :: ab : ad$.

M.—And what equality, therefore, exists between the rectangles contained by the extremes and means of their proportion?

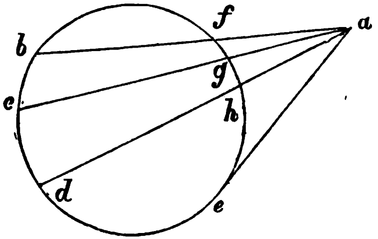
P.—The rectangle $ae, ad =$ rectangle ab, ad,—that is, rectangle $ae, ad = ab^2$.

M.—Express this equality in words.

P.—If, from a point without a circle, two straight lines be drawn, whereof one is a tangent and the other cuts the circle, the rectangle contained by the line which cuts the circle and by the part of it without the circle is equal to the *square* of the *tangent*.

M.—If, then, from the same point without a circle, several straight lines be drawn cutting the circle, what may be said of the rectangles contained by each of them and by its corresponding segment without the circle?

P.—They are equal to each other.



Draw $a\,e$, a tangent:

∴ rectangle $a\,b$, $a\,f = a\,e^2$;

and rectangle $a\,c$, $a\,g = a\,e^2$;

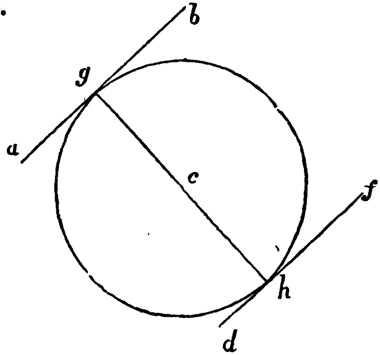
and rectangle $a\,d$, $a\,h = a\,e^2$:

∴ these rectangles are *equal* to each other.

M.—If two tangents be drawn, *not* from the same point without a circle, ascertain the manner in which they may be drawn.

P.—They may be drawn either parallel to each other, or not.

If they are parallel, the straight line which joins their points of contact passes through the centre of the circle.



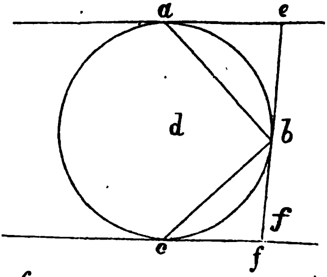
Let *a b* be parallel to *d f;*
from *c*, the centre, draw *c g* to the point of contact; and produce *g c* to *h*, a point in the tangent *d f:*
then, ∵ ∠ *c g b* is a rt. ∠,
and, ∵ *a b* is parallel to *d f,*
∴ ∠ *c h f* is likewise a rt. ∠:
and ∴ the point *h* is the point of contact of the tangent *d f.*

When the tangents are not parallel, they meet if produced, and the parts of them between the points of contact and their common point of intersection are equal to each other. (As before.)

M.—Now, determine the manner in which three tangents, from three different points, may be drawn to the circle; and ascertain the truths which result from their different relations.

P.—Only two of them may be parallel,—the third not being so.

In that case,

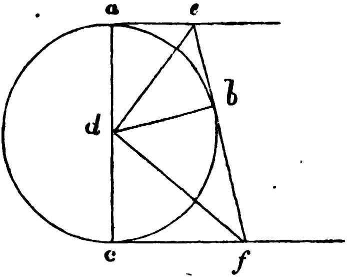


$a e + c f = a f;$
$\because a e = e b$, and $c f = f b$.
Also, ∠ *a b c* is a rt. ∠;

because a straight line joining the points, *a*, *c*, is a diameter:

∴ the arc *a c* is a semi-circumference;

and ∴ ∠ *a b c* is a right angle.



M.—But, if *d* is the centre of the circle, what kind of angle is ∠ *e d f?*

P.—It is a right angle.

Join *a c* and *d b:*

then, ∵ *a c* and *c f* are tangents, and parallel to each other;

∴ *a c* passes through the centre *d;*

also ∵ *a e* = *e b*, and *d e* is common to △s *a e d*, *b e d*,

the rt. ∠ *d a e* = rt. ∠ *d b e;*

∴ △ *a e d* = △ *b e d*, and ∴ ∠ *e d b* = ∠ *e d a;*

similarly, ∠ *f d b* = ∠ *f d c;*

and, ∴ whole ∠ *e d f* = ∠s *e d a* + *f d c:*

∴ ∠ *e d f* = ½ of 2 rt. ∠s = one rt. ∠.

If they are not parallel to each other, they will form a triangle, when produced.

M.—How would you distinguish, in words, a triangle formed by tangents drawn to the circle, from a triangle formed by chords in the circle.

P.—I would say that, the former is *circumscribed*, or, *described, about* the circle; that the latter is *inscribed in* the circle.

This paragraph, it is hoped, will suffice to show how pupils may be induced to continue their investigations of the various mutual relations of three, four, &c. tangents. The results of such inquiries are, at this stage, important so far only as they serve as means of imprinting, permanently, on their memory those geometrical truths which they have pre-established, without encumbering it with an additional weight. The investigation of the mutual relations of circumscribed and inscribed figures is, for learners so young, frequently too prolix, and, therefore, not of a kind that may be continued with sustained interest. Far more important, moreover, are the truths resulting from a combination of several circles, — the study of which is, the method of constructing Rectilinear Figures.

SUBSTANCE OF SECTION III.

1. If, from a point without a circle, straight lines be drawn cutting the circle, the greatest is that which passes through the centre; and of the others, that which is nearer to the one in which is the centre, is less than one more remote.

2. Also, a pair of equal straight lines may be drawn to a circle from the same point without the circle, one on each side of the greatest line.

3. If, from a point without a circle, straight lines be

drawn to meet the circumference, the least is that which, when produced, passes through the centre; and of the others, that which is farther from the least line is greater than one nearer to it.

4. The straight line joining the centre of a circle and the point of contact of a tangent is at right angles to the tangent.

5. The straight line joining any two points in the circumference of a circle is within the circle.

6. A tangent touches a circle in one point only.

7. If, from the point of contact of a tangent, a straight line be drawn cutting the circle, the angles which it makes with the tangent are equal to the angles upon the adjacent arcs.

8. From the point of contact of a tangent, no straight line can be drawn between the tangent and the circumference which does not cut the circle.

9. Two tangents from the same point, without a circle, are equal to each other.

10. If, from a point without a circle, two straight lines be drawn whereof one is a tangent and the other cuts the circle, the rectangle contained by the line which cuts the circle and by the part of it without the circle is equal to the square of the tangent.

11. If, from the same point without a circle, several straight lines are drawn cutting the circle, the rectangles contained by each of them and by its corresponding part without the circle are equal to each other.

12. If two tangents are parallel to each other, the straight line joining their points of contact passes through the centre.

SECTION IV.

TWO CIRCLES.

The object of this paragraph is the investigation of such truths as arise from the various combinations of two circles.

The order in which the branches of this subject naturally present themselves to the contemplation of the mind is the following: first—*two circles touching each other;* next, *two circles as intersecting each other;* and, thirdly, *the results of these investigations as applicable,* in combination with those of the preceding paragraphs, *to the solution of problems.*

M.—What may be said of the equality of two circles, generally?

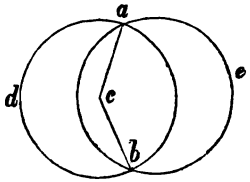
P.—Two circles may be either equal to each other, or they may not be equal to each other.

M.—Ascertain the various ways in which two equal circles may be applied to each other.

P.—The centre of the one may be put upon the centre of the other; and then, because the circles are equal to each other, their circumferences must coincide.

If the centre of one of the circles be *not* put upon the centre of the other, their circumferences may cut each other, or touch each other, or, neither cut nor touch each other.

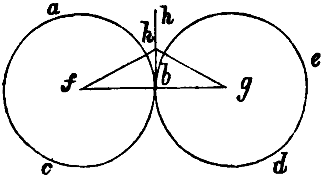
If two circles cut each other, they cannot cut each other in more than two points.



Let *c* be the centre of the circles *a d b*, *b e a;* and let *a*, *b*, be two points of intersection:

∴ *c a* = *c b*, and, from *c*, no other straight line equal to *c a* or *c b* can be drawn to the circumference of the circle *a e b*, unless the point *c* is, likewise, the centre of the circle *a e b*.

If two circles touch each other, they can have only *one* point of contact.



Let *b* be a point of contact of the circles *a b c* and *b e d;*

draw *b f* to the centre,

and *b h* at rt. ∠s to *b f*,

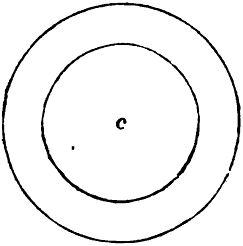
and produce *f b:*

∴ ∠s *h b f* and *h b g* are rt. ∠s,

and ∴ *b g* passes through the centre of the circle *b h e*. And, these circles cannot have another point of contact besides the point *b;* because *f k* + *k g* is greater than *f g*.

From this it follows, that, if two circles touch each other externally, the straight line joining their centres passes through the point of contact.

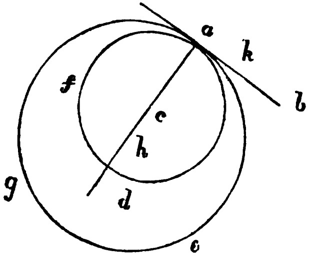
If two unequal circles neither cut nor intersect each other, they may be so placed that the centre of the one shall be upon the centre of the other.*



M. Well,—such circles are called *concentric* circles.

P.—Or, they may be so placed that their centres are *not* the one upon the other.

If two unequal circles touch each other internally, they can have only one point of contact.



Let *a* be a point of contact;

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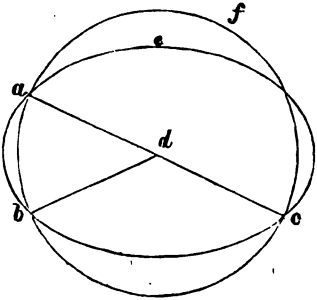
* Here and in other places, copious answers or statements are set-down as flowing, apparently, from *one* question, though actually, perhaps, resulting from several queries, hints, and helps,—the insertion of which would, to the discerning and competent instructor, be both tedious and useless.

draw *a c* to the centre of the circle *a d f*,
draw *a b* at right angles to *a c*,
and produce *a c :*
then ∵ ∠ *b a c* is a rt. ∠,
∴ *b a* touches the circles *a f d* and *a e g*,
and ∴ *a c*, produced, passes, likewise, through the centre, *h*, of the circle *a e g*.

Moreover, these circles cannot have another point of contact besides the point *a;* for, a straight line drawn from any other point *k*, in *a b*, through *c*, cannot likewise pass through the point *m*.

Hence it follows, that, if two circles touch each other internally, the straight line joining their centres passes through the point of contact.

Obs.—Most of the preceding truths may be considered almost as axioms; and, like all axioms, they may be clearly established, by assuming the contrary to be true, and, then, showing the absurdity resulting from the supposition. In order to show, for instance, that two circles, which cut each other, cannot cut each other in more than two points, the contrary may be assumed,—namely,



that the circumference *a b c e can* cut the circumference *a b c f* in more than two points,—in *a, b, c.*

Let *d* be the centre of the circle *a b c f*,
and join *d a, d b, d c ;*
$\therefore d a = d b = d c$,
and ∴ the point *d* is, likewise, the centre of the circle *a b c e :*
but, two circles which *cut each other* cannot have the *same* centre;
therefore, the assumption (that two circles *can* cut each other in more than two points,) is erroneous; and, since it is an error to assume any number of points *except two*, it follows, that two circles can cut each other in two points only.

It may be, here, remarked that, *as an useful exercise*, the master should require the pupils, to draw lines, in certain directions, in circles which intersect or touch each other, and, then, to discover such truths as necessarily follow therefrom.

An exercise of greater importance, however, is the application of the properties of the circle to the construction of rectilineal figures. The best course of proceeding, in this and every case, is that in which the subject *naturally* presents itself, of which the following outlines are offered for the guidance of the master.

I.—STRAIGHT LINES.

It is required,

1. To draw a straight line equal to a given straight line.

2. To draw two, three, four, &c. straight lines equal to a given straight line.

3. To bisect a given finite straight line.

4. To draw a straight line at right angles to a given straight line.

5. To draw a straight line at right angles to a given straight line from a given point in the same.

6. To draw a straight line at right angles to a straight line from a given point without it.

7. To draw a straight line parallel to a given straight line.

8. From a given point, to draw a straight line parallel to a given straight line.

II.—ANGLES.

It is required,

1. To make a right angle.
2. To make an obtuse angle.
3. To make an acute angle.
4. To make an angle equal to a given angle.
5. To bisect a given angle.

III.—TRIANGLES.

It is required,

1. To describe a right-angled triangle.
2. To describe an obtuse-angled triangle.
3. To describe an acute-angled triangle.

4. To make a given finite straight line the side opposite to the right angle,* in a right-angled triangle.

5. To make a given finite straight line one of the sides containing the right angle, in a right-angled triangle.

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* Hypotenuse [ὑποτείνουσα, Gr.].

6. To describe an equilateral triangle.

7. Upon a given finite straight line, to describe an equilateral triangle.

8. To describe an isosceles triangle.

9. To make a given finite straight line the base of an isosceles triangle.

10. What requisites must three straight lines have, in order to become the sides of a triangle?

11. With three given straight lines, (under the restriction in question,) to construct a triangle.

IV.—QUADRILATERAL FIGURES.

It is required,

1. To describe a square.

2. To describe a rhomb.

3. To describe a rectangle.

4. To describe a parallelogram.

5. Upon a given finite straight line, to describe a square.

6. To make a given finite straight line the diagonal of a square.

7. To make a given straight line the diagonal of a rhomb,—of a rectangle,—of a parallelogram.

V.—POLYGONS.

It is required,

1. To describe a pentagon [not an equilateral and equiangular pentagon].*

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* This problem is proposed, now, in order to stimulate inquiry: it solution should be learnt from Euclid.

2. To describe a regular hexagon.

3. Upon a given straight line, to describe a regular hexagon.

VI.—CIRCLES.

It is required,

1. To find the centre of a given circle.

2. [What requisite must a given straight line have in order to be a chord in a given circle?]

3. With the restriction in question, to place a given straight line in a given circle.

4. To place a right angle in a given circle so that its vertex may be in the circumference of the circle.

5. To describe a circle about a right-angled triangle.

6. To describe a circle about an obtuse-angled triangle.

7. To describe a circle about an acute-angled triangle.

8. At a given point in the circumference of a circle, to draw a tangent.

9. From a given point without a circle, to draw a tangent.

10. In a right-angled, obtuse-angled, and acute-angled triangle, to inscribe a circle.

11. In a given circle, to inscribe a square.

12. About a given circle, to describe a square.

13. In a given square, to inscribe a circle.

14. About a given square, to describe a circle.

15. In a given circle, to inscribe a rectangle.

16. [Can a rectangle be described about a given circle?]

17. [Can a rhomb be inscribed or described about a given circle?]

18. [Under what restrictions can a quadrilateral figure be inscribed in a given circle?]

19. To inscribe a circle in a given regular pentagon.

20. To describe a circle about a given regular pentagon.

21. In a given circle, to inscribe a regular hexagon.

22. About a given circle, to describe a regular hexagon.

23. In a given regular hexagon, to inscribe a circle.

24. About a given regular hexagon, to describe a circle.

If care be taken to follow the order here adopted, the pupils will not experience many or great difficulties, in finding-out the solutions of these problems. It is obvious, that an almost endless variety of seasonable exercises may be devised; and, to stimulate inquiry and beget a liking for mathematical pursuits, the pupils should, frequently, be encouraged to challenge each other to the solution of problems of their *own* proposal or selection.

Having proceeded through this rudimental course, the pupils of Cheam-school commence the study of Euclid's Elements. The definitions, axioms, and postulates of Book I. which have been already fully discussed in the course of these preparatory lessons, are now dictated, seriatim, to the pupils, and committed to memory. The first problem is next submitted to the class, to be worked out by them, and the pupil who first succeeds in finding the demonstration, is desired to advance to the school-slate, and pronounce it aloud to his class-fellows. These, again, each in his turn, give an *oral* demonstration,—which they then write out, at length, on their own slates,—the master correcting the errors and supplying the defects. After having proceeded, in this manner, through Book I. II. III. IV. V. VI. XI., each pupil is provided with a copy of the *Elements of Euclid,* that he may repeat the course, and become thoroughly familiar with the elements of the science. For this purpose a certain portion of time is set apart; but he still carries forward his Mathematical studies in the departments of Algebra, Trigonometry, Conic Sections, &c.—To these studies he brings a mind thoroughly versed in all the preceding steps, trained to Mathematical inquiry, and habituated to the processes of Mathematical demonstration.

THE END.

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